WILD TRIANGLES IN 3-CONNECTED MATROIDS

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ABSTRACT. Tutte's Triangle Lemma proves that if $\{a, b, c\}$ is a triangle in a 3-connected matroid and neither $M\backslash a$ nor $M\backslash b$ is 3-connected, then M has a triad that contains a and exactly one of b and c. Hence $\{a, b, c\}$ is contained in a fan of M with at least four elements. In this paper we ask for a somewhat stronger conclusion. When is it that, for each t in $\{a, b, c\}$, either $M\backslash t$ is not 3-connected, or $M\backslash t$ has a 3-separation that is not equivalent to a 3-separation induced by M? The main result describes the structure of M relative to $\{a, b, c\}$ when this occurs. This theorem generalizes a result of Geelen and Whittle for sequentially 4-connected matroids. The motivation for proving this result was for use as an inductive tool for connectivity results aimed at representability questions. In particular, Geelen, Gerards, and Whittle use it in their proof of Kahn's Conjecture for 4-connected matroids.

1. INTRODUCTION

In the study of inequivalent representations of matroids, it is increasingly clear that one has to work with a variety of connectivity notions that are intermediate between 3- and 4-connectivity. More precisely, one studies 3-connected matroids whose 3-separations are controlled in some way. Two specific examples of this type of connectivity are the notions of sequential 4-connectivity and k-coherence, introduced in [3] and [4], respectively. To enable inductive arguments to be made within the class of matroids with a given connectivity, it would be ideal to have analogues of classical tools for 3-connectivity such as Tutte's Triangle Lemma [10], Tutte's Wheels-and-Whirls Theorem [10], and Seymour's Splitter Theorem [9].

In this paper we consider an extension of Tutte's Triangle Lemma but, rather than focus on a *particular* connectivity notion, we consider a more general question. *Given a triangle T in a 3-connected matroid M, when is it impossible to delete an element from T without either losing 3-connectivity or creating new unwanted 3-separations?* The main result of this paper, Theorem 1.1, answers this question by describing the structure of the matroid relative to such a triangle.

By studying the problem at this level of generality, we obtain triangle theorems for particular notions of connectivity as corollaries. For example,

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the triangle theorem for internally 4-connected matroids [3, Theorem 6.1] is a corollary of Theorem 1.1 as is the triangle theorem for k-coherent matroids [4]. Indeed, the proof of the triangle theorem for k-coherent matroids given in [4] uses the result of this paper. Moreover, the triangle theorem for k-coherent matroids forms part of the connectivity theory that leads to a proof of Kahn's Conjecture [5] for 4-connected matroids [4].

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X|, |E - X| \geq k$. A k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$, and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1)$ or $\mathrm{fcl}(B_1)$ is $E(M)$, then (A_1, B_1) is *sequential.*

Let e be an element of a matroid M such that both M and $M\backslash e$ are 3-connected. A 3-separation (X, Y) of $M\backslash e$ is *well blocked* by e if, for all exactly 3-separating partitions (X', Y') equivalent to (X, Y), neither $(X' \cup e, Y')$ nor $(X', Y' \cup e)$ is exactly 3-separating in M. An element f of M *exposes* a 3-separation (U, V) if (U, V) is a 3-separation of $M\backslash f$ that is well blocked by f. Although (U, V) is actually a 3-separation of $M\backslash f$, we shall often say that *f exposes a 3-separation (U, V) in M.* Evidently, if e exposes an exactly 3-separating partition (E_1, E_2), then e exposes all exactly 3-separating partitions (E_1', E_2') that are equivalent to (E_1, E_2).

With the above technicalities in hand, we define a triangle T of a 3-connected matroid M to be *wild* if, for all t in T, either $M\backslash t$ is not 3-connected, or $M\backslash t$ is 3-connected and t exposes a 3-separation in M. The task of this paper is to characterize wild triangles.

We begin by describing some particular examples. An ordered partition $(P_1, P_2, \ldots, P_n)$ of the ground set of a 3-connected matroid M is a *flower* if $\lambda_M(P_i) = 2 = \lambda_M(P_i \cap P_{i+1})$ for all i in $\{1, 2, \ldots, n\}$ where all subscripts are interpreted modulo n. A *quad* is a 4-element set in M that is both a circuit and a cocircuit. In particular, a quad is 3-separating. In describing these examples, we shall use some technical language for flowers. This terminology, which was developed in [7], is recalled in Section 2. In the matroid M illustrated in Figure 1, $M\backslash a, b, c$ has a tight swirl-like flower $(P_1, P_2, \ldots, P_6)$. Moreover, $a \in \mathrm{cl}(P_1 \cup P_2) \cap \mathrm{cl}(P_3 \cup P_4 \cup P_5 \cup P_6)$, and b and c are symmetrically placed. Other wild triangles can be obtained by modifying this situation. For example, the underlying flower $(P_1, P_2, \ldots, P_6)$ need not be swirl-like,

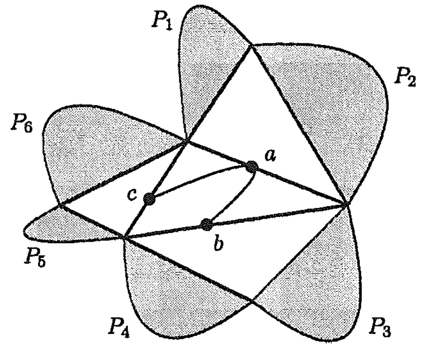

FIGURE 1. A standard wild triangle $\{a, b, c\}$.

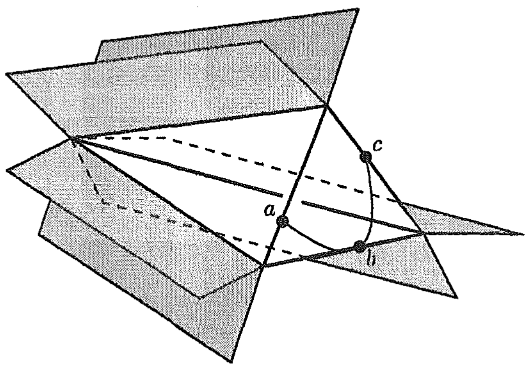

FIGURE 2. A costandard wild triangle $\{a, b, c\}$.

but may be spike-like or a copaddle (but never a paddle). Moreover, one can replace certain elements of the flower $(P_1, P_2, \ldots, P_6)$ by series classes, but only in a controlled way. We will say that a wild triangle of the types described above is a *standard* wild triangle. A precise definition of standard wild triangles is given in Section 2.

Via a $\Delta - Y$ exchange, we can obtain another type of wild triangle. Let T be a standard wild triangle of the matroid M, and let M' be the matroid obtained by first performing a $\Delta - Y$ exchange on the triangle T and then taking the dual. It is readily checked that the triangle corresponding to T in M' is a wild. We call a wild triangle obtained in this way a *costandard* wild triangle. Figure 2 illustrates a costandard wild triangle.

Let M be a matroid with a non-sequential 3-separation (X, Y). Then X is a *trident* in M if $|X| = 7$, say $X = \{a, b, c, t, s, u, v\}$, and $\{a, b, c\}$ is a triangle, while $\{t, s, u, b\}$, $\{t, u, v, c\}$, and $\{t, s, v, a\}$ are quads exposed in

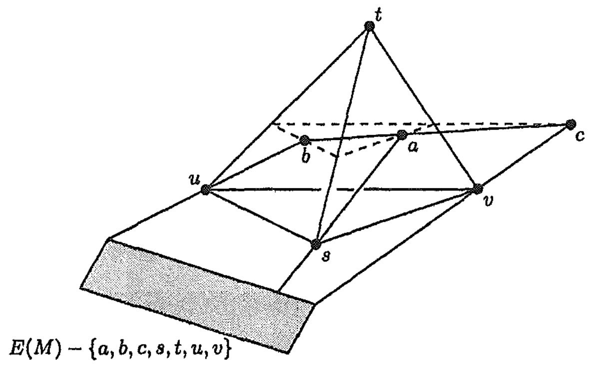

FIGURE 3. A trident.

$M\backslash a$, $M\backslash b$, $M\backslash c$, respectively. Figure 3 illustrates a trident. Evidently, a trident contains a wild triangle that is neither standard nor costandard.

Our final example of a wild triangle is well known. In a 3-connected matroid, if F is a fan with at least four elements and T is a triangle in F that contains neither end of F, then T is an *internal triangle* of F. If T is such a triangle, then T is wild since, for all t in T, the matroid $M\backslash t$ is not 3-connected. At last, we can state our main theorem.

Theorem 1.1. *Let T be a wild triangle of a 3-connected matroid M with at least twelve elements. Then T is either a standard wild triangle, a costandard wild triangle, a triangle in a trident of M, or an internal triangle of a fan of M.*

Let M be a 3-connected matroid. If M has no 3-separations (X, Y) with $|X|, |Y| \geq 4$, then M is *internally 4-connected*; M is *sequentially 4-connected* if it has no non-sequential 3-separations. It is easily seen that no matroid with at least 12 elements and a wild triangle is internally 4-connected, so an immediate consequence of Theorem 1.1 is the following result, which establishes the substantial part of [3, Theorem 6.1].

Corollary 1.2. *Let T be a triangle of an internally 4-connected matroid M where $|E(M)| \geq 12$. Then there is an element t in T such that $M\backslash t$ is sequentially 4-connected.*

An overview of the paper follows. The next section presents some terminology along with some basic lemmas that we shall use. In Section 3, we give precise definitions of the types of wild triangles. Then we state Theorem 3.1, a strengthening of Theorem 1.1, and we present one other result, Corollary 3.2, which gives more detailed information about the structure around a wild triangle. That section concludes with Corollary 3.3, which gives a quick way to differentiate between the four types of wild triangles. In Section 4, we prove an extension of Tutte's Triangle Lemma. This result splits the proof of Theorem 1.1 into two cases and completely settles one of the cases. The rest of the paper is concerned with settling the proof in the other case. Section 5 sets up the start of the proof and establishes some basic facts that will be used throughout the proof. In Section 6, we give an overview of the structure of the rest of the proof, dividing the remaining argument into six cases, (A)-(F). Section 7 shows that, in each of the first four of these cases, $\{a, b, c\}$ is in a trident in M. In Section 8, we consider case (F). We show that, by performing a $\Delta - Y$ exchange on M and dualizing, we can reduce to the subcase of case (E) in which we have symmetry between a, b, and c. In Section 9, we show that, when case (E) occurs, either one of cases (A)-(D) occurs, or we are in the subcase of (E) in which we have symmetry between a, b, and c. That section concludes by completing the proofs of Theorems 1.1 and 3.1 and of Corollary 3.3. Finally, Section 10 proves Corollary 3.2.

2. Preliminaries

The matroid terminology used here will follow Oxley [6] except that the simplification and cosimplification of a matroid N will be denoted by $\mathrm{si}(N)$ and $\mathrm{co}(N)$, respectively. We shall write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. This paper will use some results and terminology from our papers describing the structure of 3-separations in 3-connected matroids [7, 8]. In this section, we introduce the relevant definitions. We also prove some elementary connectivity results that will be used in the proof of the main theorem.

Let X be an exactly 3-separating set of M. If there is an ordering $(x_1, x_2, \ldots, x_n)$ of X such that, for all i in $\{1, 2, \ldots, n\}$, the set $\{x_1, x_2, \ldots, x_i\}$ is 3-separating, then X is *sequential.* An exactly 3-separating partition (X, Y) of M is *sequential* if either X or Y is a sequential 3-separating set.

The connectivity function λ_M of a matroid M has a number of attractive properties. For example, $\lambda_M(X) = \lambda_M(E - X)$. Moreover, the connectivity functions of M and its dual M^* are equal. To see this, it suffices to note the easily verified fact that

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

We shall often abbreviate λ_M as λ.

One of the most useful features of the connectivity function of M is that it is submodular, that is, for all $X, Y \subseteq E(M)$,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y).$$

This means that if X and Y are k-separating, and one of $X \cap Y$ or $X \cup Y$ is not $(k-1)$-separating, then the other must be k-separating. The next two lemmas specialize this fact to 3-connected and 2-connected matroids, respectively.

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

The last lemma will be in constant use throughout the paper. For convenience, we use the phrase *by uncrossing* to mean "by an application of Lemma 2.1."

Lemma 2.2. *Let M be a 2-connected matroid, and let X and Y be subsets of $E(M)$ with $\lambda(X) = 2$ and $\lambda(Y) = 1$. If neither $X \cap Y$ nor $E - (X \cup Y)$ is empty, then $\lambda(X \cap Y) = 1$ or $\lambda(X \cup Y) = 1$.*

The connectivity function is also well known to be monotone under taking minors.

Lemma 2.3. *Let X be a set in a matroid M. If N is a minor of M, then*

$$\lambda_N(X \cap E(N)) \leq \lambda_M(X).$$

Delta-Wye Exchange. Let Δ be a triangle $\{a, b, c\}$ of a matroid M and consider a copy of $M(K_4)$ that has Δ as a triangle and has $\{a', b', c'\}$ as the complementary triad, where e' is the element of $M(K_4)$ that is not in a triangle with e. Let $P_\Delta(M(K_4), M)$ be the *generalized parallel connection* of $M(K_4)$ and M, that is, the matroid on $E(M(K_4)) \cup E(M)$ whose flats are those subsets X of $E(M(K_4)) \cup E(M)$ such that $X \cap E(M(K_4))$ is a flat of $M(K_4)$ and $X \cap E(M(K_4))$ is a flat of M. We shall denote the matroid $P_\Delta(M(K_4), M)\backslash\Delta$ by ΔM and say that this matroid has been obtained from M by a $\Delta - Y$ *exchange on* Δ. We observe that ΔM has ground set $(E(M) - \{a, b, c\}) \cup \{a', b', c'\}$. It is common to relabel a', b', and c' as a, b, and c so that M and ΔM have the same ground set. For example, we do this in the next section. However, in Section 8, when we are proving various properties of ΔM, we shall retain the original labelling.

Fans. Let S be a subset of a 3-connected matroid M. We call S a *fan* of M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all i in $\{1, 2, \ldots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad; and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

The ordering $(s_1, s_2, \ldots, s_n)$ is called a *fan ordering* of S. When $n \geq 4$, the elements s_1 and s_n are the only elements of the fan that are not in both a triangle and a triad contained in S. We call these elements the *ends* of the fan S. The remaining elements of S are the *internal elements* of the fan. An *internal triangle* of S is a triangle all of whose elements are internal elements of S.

Flowers. The notion of a flower was introduced in [7] to deal with crossing 3-separations, that is, 3-separations (A_1, A_2) and (B_1, B_2) for which each of the intersections $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, and $A_2 \cap B_2$ is non-empty. When each of these intersections has at least two elements, Lemma 2.1 implies that each is exactly 3-separating. Moreover, the union of any consecutive pair in the cyclic ordering $(A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_2, A_2 \cap B_1)$ is exactly 3-separating. This 4-tuple is an example of a flower.

Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ in a 3-connected matroid M. The sets $P_1, P_2, \ldots, P_n$ are the *petals* of Φ. Each has at least two elements. It is shown in [7, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \ldots, P_n)$. Observe that when $n \leq 3$, the concepts of an anemone and a daisy coincide but, for $n \geq 4$, a flower cannot be both an anemone and a daisy.

Equivalent flowers, and tight and loose petals. Let Φ_1 and Φ_2 be flowers of a matroid M. A natural quasi ordering on the collection of flowers of M is obtained by setting $\Phi_1 \preceq \Phi_2$ whenever every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$, we say that Φ_1 and Φ_2 are *equivalent* flowers of M. Hence equivalent flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. The *order* of a flower Φ is the minimum number of petals in a flower equivalent to Φ.

Let Φ be a flower of M. An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ. An element that is not loose is *tight*. We say that a petal P_i is *loose* if all elements in P_i are loose. A *tight* petal is one that is not loose, that is one that contains at least one tight element. In fact, it is not difficult to show that a tight petal must contain at least two tight elements.

Local connectivity and flower types. The classes of anemones and daisies can be further refined using the concept of local connectivity. For sets X and Y in a matroid M, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined to be

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

When M is $\mathbb{F}$-representable and hence viewable as a subset of the vector space $V(r(M), \mathbb{F})$, the local connectivity $\sqcap(X, Y)$ is precisely the rank of the intersection of those subspaces in $V(r(M), \mathbb{F})$ that are spanned by X and Y.

For $n \geq 3$, an anemone $(P_1, P_2, \ldots, P_n)$ is called

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in \{1, 2, \ldots, n\}$;
(ii) a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct $i, j \in \{1, 2, \ldots, n\}$; and
(iii) *spike-like* if $n \geq 4$, and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

Similarly, a daisy $(P_1, P_2, \ldots, P_n)$ is called

(i) *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and
(ii) *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. It is shown in [7] that every flower with at least three petals is one of these six different *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous.

The rest of this section contains some elementary lemmas that will be frequently used throughout the paper.

Lemma 2.4. *Let M be a 3-connected matroid. If f exposes a 3-separation (U, V) in M, then (U, V) is non-sequential. In particular, $|U|, |V| \geq 4$. Moreover, if $|V| = 4$, then V is a quad of $M \backslash f$.*

Proof. If (U, V) is sequential, then, without loss of generality, $(U, V) \cong (U', \{v_1, v_2\})$, an exactly 3-separating partition of $M \backslash f$. Since $(U' \cup f, \{v_1, v_2\})$ is an exactly 3-separating partition of M, we deduce that (U, V) is not well blocked by f, so f does not expose (U, V).

Now suppose that $|V| = 4$. Since U does not span V in M or M^*, we have $r(V), r^*(V) \geq 3$. As $r(V) + r^*(V) - |V| = 2$, we deduce that $r(V) = r^*(V) = 3$. If V contains a triangle, then V is sequential. Hence V is a circuit. By duality, we conclude that V is a quad. $\square$

Lemma 2.5. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M. Suppose $|X| \geq 3$ and $x \in X$. Then*

(i) *$x \in \mathrm{cl}^{(*)}(X - x)$, that is, $x \in \mathrm{cl}(X - x)$ or $x \in \mathrm{cl}^*(X - x)$; and*
(ii) *$(X - x, Y \cup x)$ is exactly 3-separating if and only if x is in exactly one of $\mathrm{cl}(X - x) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - x) \cap \mathrm{cl}^*(Y)$.*

Proof. We shall prove (i) and omit the similar proof of (ii). We have

$$r(X) + r^*(X) - |X| = 2 \leq r(X - x) + r^*(X - x) - |X - x|$$

where the last inequality holds because M is 3-connected. Thus

$$r(X-x)+r^*(X-x)-|X| \leq r(X)+r^*(X)-|X| \leq r(X-x)+r^*(X-x)-|X|+1.$$

Hence either $r(X - x) = r(X)$ or $r^*(X - x) = r^*(X)$. Thus either $x \in \mathrm{cl}(X - x)$ or $x \in \mathrm{cl}^*(X - x)$. $\square$

The elementary proof of the next lemma is omitted.

Lemma 2.6. *For a matroid M, let (X, Y) be a k-separation of $M \backslash T$ and $\{T_X, T_Y\}$ be a partition of T into possibly empty sets. If $T_X \subseteq \mathrm{cl}(X)$ and $T_Y \subseteq \mathrm{cl}(Y)$, then $(X \cup T_X, Y \cup T_Y)$ is a k-separation of M.*

Lemma 2.7. *Let e be an element of a matroid M and X be a subset of $E(M) - e$. If $\lambda(X) = k$ and $\lambda(X \cup e) \leq k - 1$, then $e \in \mathrm{cl}(X)$ and $e \in \mathrm{cl}^*(X)$.*

Proof. We have $k = r(X) + r^*(X) - |X|$ and $k - 1 \geq r(X \cup e) + r^*(X \cup e) - |X \cup e|$. Hence

$$r(X \cup e) + r^*(X \cup e) - |X| \leq k = r(X) + r^*(X) - |X|,$$

so $r(X \cup e) = r(X)$ and $r^*(X \cup e) = r^*(X)$. Thus $e \in \mathrm{cl}(X)$ and $e \in \mathrm{cl}^*(X)$. □

Lemma 2.8. *Let $\{a, b, c\}$ be a triangle of a matroid M and suppose that M and $M \backslash a$ are 3-connected. Let (A_1, A_2) be a 3-separation of $M \backslash a$ that is exposed by a. Then*

(i) *neither A_1 nor A_2 contains $\{b, c\}$; and*
(ii) *if $b \in A_1$, then $b \in \mathrm{cl}_{M \backslash a}(A_1 - b)$.*

Proof. If $\{b, c\} \subseteq A_1$, then $a \in \mathrm{cl}(A_1)$, so $(A_1 \cup e, A_2)$ is a 3-separation of M; a contradiction. Hence, by symmetry, (i) holds. Now suppose that $b \in A_1$. By Lemma 2.4, $|A_1|, |A_2| \geq 4$. Assume $b \notin \mathrm{cl}_{M \backslash a}(A_1 - b)$. Then $r(A_1 - b) + r(A_2 \cup b) \leq r(A_1) + r(A_2)$. Since $|A_1 - b| \geq 3$, it follows that $(A_1 - b, A_2 \cup b)$ is a 3-separating partition equivalent to (A_1, A_2). But $a \in \mathrm{cl}(A_2 \cup b)$ so (A_1, A_2) is not well blocked by a. □

The proof of the next result is straightforward; we omit the details.

Lemma 2.9. *Let Q be a quad in a 3-connected matroid M. If $e \in Q$, then $\mathrm{si}(M/e)$ is 3-connected.*

The final result of the section is an elementary property of flowers.

Lemma 2.10. *Let $(P_1, P_2, \ldots, P_k)$ be a flower in a 3-connected matroid. If P_2 is loose, then*

$$P_2 \subseteq \mathrm{fcl}(P_1).$$

Proof. We argue by induction on $|P_2|$. Suppose that $P_2 = \{x, y\}$ and x is in $\mathrm{cl}^{(*)}(P_i) - P_i$ for some $i \neq 2$. If $i = 1$, then Lemma 5.2 of [7] implies that $y \in \mathrm{cl}^{(*)}(P_1 \cup x)$, so $P_2 \subseteq \mathrm{fcl}(P_1)$, as required. If $i \neq 1$, then $P_3 \cup P_4 \cup \cdots \cup P_k \cup x$ is 3-separating. Hence so is $P_1 \cup y$, and Lemma 5.2 of [7] again implies that $P_2 \subseteq \mathrm{fcl}(P_1)$.

Now assume the result holds for $|P_2| < n$ and let $|P_2| = n \geq 3$. As P_2 is loose, it has an element x such that $x \in \mathrm{cl}^{(*)}(P_i) - P_i$ for some $i \neq 2$. If $i = 1$, then $(P_1 \cup x, P_2 - x, P_3 \ldots, P_k)$ is a flower in which $P_2 - x$ is loose so, by the induction assumption, $P_2 - x \subseteq \mathrm{fcl}(P_1 \cup x)$. Since x is in $\mathrm{fcl}(P_1)$, we deduce that $P_2 \subseteq \mathrm{fcl}(P_1)$. Now suppose that $i \neq 1$. Then $(P_1, P_2 - x, P_3 \ldots, P_i \cup x, \ldots, P_k)$ is a flower in which $P_2 - x$ is loose. Hence, by by the induction assumption, $P_2 - x \subseteq \mathrm{fcl}(P_1)$. Moreover, as both $P_2 - x$ and P_2 are 3-separating, $x \in \mathrm{cl}^{(*)}(P_2 - x)$. Hence $x \in \mathrm{fcl}(P_1)$ and so $P_2 \subseteq \mathrm{fcl}(P_1)$. The lemma follows by induction. □

3. Wild Triangles

In this section we give more precise definitions of the types of wild triangles. We then state a theorem that is a strengthening of Theorem 1.1 in that it gives additional information about the structure of a matroid M around a wild triangle. Finally, we state three corollaries that give yet more structural information about wild triangles.

Let $\{a, b, c\}$ be a triangle of a 3-connected matroid M. Then $\{a, b, c\}$ is a *standard* wild triangle if there is a partition $\mathbf{P} = (P_1, P_2, \ldots, P_6)$ of $E(M) - \{a, b, c\}$ such that $|P_i| \geq 2$ for all i and the following hold.

(i) $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, $M\backslash a, b, c$ is connected, and $\mathrm{co}(M\backslash a, b, c)$ is 3-connected.
(ii) $(P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup c)$ is a flower in M.
(iii) $(P_2 \cup P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup P_1 \cup c)$, $(P_4 \cup P_5 \cup P_6 \cup c, P_1 \cup P_2 \cup P_3 \cup a)$, and $(P_6 \cup P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup P_5 \cup b)$ are 3-separations exposed in M by a, b, and c, respectively.

A partition $\mathbf{P}$ satisfying these conditions is a partition *associated* to $\{a, b, c\}$.

Note that a partition associated to a wild triangle need not be unique, even up to equivalence. But such a partition suffices to localize the structure of the triangle relative to the rest of the matroid.

Let Δ be a triangle $\{a, b, c\}$ of a 3-connected matroid M and let ΔM denote the matroid obtained by performing a $\Delta - Y$ exchange on Δ. We assume that the ground sets of M and ΔM are equal by labelling the latter in the natural way. Then Δ is a *costandard* wild triangle in M if Δ is a standard wild triangle in $(\Delta M)^*$. Let $\mathbf{P} = (P_1, P_2, \ldots, P_6)$ be a partition of $E(M) - \{a, b, c\}$. Then $\mathbf{P}$ is *associated* to the costandard wild triangle Δ in M if $\mathbf{P}$ is associated to the standard wild triangle Δ in $(\Delta M)^*$.

Let R be a 3-separating set $\{a, b, c, s, t, u, v\}$ in a 3-connected matroid M, where $\{a, b, c\}$ is a triangle. Then R is a *trident* with wild triangle $\{a, b, c\}$ if $\{t, s, u, b\}$, $\{t, u, v, c\}$, and $\{t, s, v, a\}$ are exposed quads in $M\backslash a$, $M\backslash b$, and $M\backslash c$, respectively. These quads need not be the only 3-separations exposed by a, b, or c. A more complete discussion of the 3-separations exposed by wild triangles in tridents is given in Section 7. Observe that $(M/t)|(R-t) \cong M(K_4)$.

Theorem 3.1. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M, where $|E(M)| \neq 11$, and suppose that $\{a, b, c\}$ is not an internal triangle of a fan of M. Then $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. Moreover, if $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) are 3-separations exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1$, $b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$, then exactly one of the following holds:*

(i) *$\{a, b, c\}$ is a wild triangle in a trident;*
(ii) *$\{a, b, c\}$ is a standard wild triangle and $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) can be replaced by equivalent 3-separations such that*

(a) $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ *is a partition associated to* $\{a, b, c\}$*;*

(b) *every* 2*-element cocircuit of* $M \backslash a, b, c$ *meets exactly two of* $A_2 \cap B_1, B_2 \cap C_1$*, and* $C_2 \cap A_1$*; and*

(c) *in* $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$*, every union of consecutive sets is exactly* 3*-separating in* $M \backslash a, b, c$*;*

(iii) $\{a, b, c\}$ *is a costandard wild triangle; more particularly, if* M' *is the matroid that is obtained from* M *by performing a* $\Delta - Y$ *exchange on* $\{a, b, c\}$ *in* M *and then taking the dual of the result, then* M' *is* 3*-connected and* $((A_2 - c) \cup b, (A_1 - b) \cup c), ((B_2 - a) \cup c, (B_1 - c) \cup a)$*, and* $((C_2 - b) \cup a, (C_1 - a) \cup b)$ *are* 3*-separations in* M' *exposed by* a, b*, and* c*, respectively. Moreover, (ii) holds when* $(M, A_1, A_2, B_1, B_2, C_1, C_2)$ *is replaced by* $(M', (A_2 - c) \cup b, (A_1 - b) \cup c, (B_2 - a) \cup c, (B_1 - c) \cup a, (C_2 - b) \cup a, (C_1 - a) \cup b)$*.*

The next corollary gives a more detailed description of the structure associated with a standard wild triangle. We omit such a description for costandard wild triangles, but note that their detailed structure can be obtained straightforwardly be considering the effect of performing a $\Delta - Y$ exchange on a standard wild triangle and dualizing. Let $(P_1, P_2, \ldots, P_n)$ be a partition $\mathbf{P}$ of a set E and let A be a subset of E. Then the partition of Z *induced* by $\mathbf{P}$ is the partition $(P_1 \cap Z, P_2 \cap Z, \ldots, P_n \cap Z)$.

Corollary 3.2. *Let* $\{a, b, c\}$ *be a standard wild triangle of a* 3*-connected matroid* M *with an associated partition* $\mathbf{P}$*. Let* $N = \mathrm{co}(M \backslash a, b, c)$ *and let* $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_6)$ *be the partition of* $E(N)$ *induced by* $\mathbf{P}$*. Then* $\mathbf{Q}$ *is a tight flower in* N *that is swirl-like, spike-like, or a copaddle, and the following hold.*

(i) *If* $\mathbf{Q}$ *is swirl-like, then the non-trivial series classes of* $M \backslash a, b, c$ *have size exactly* 2 *and there are at most three such series pairs. An element of* $E(N)$ *corresponding to a series pair of* $M \backslash a, b, c$ *is in one of* $\mathrm{cl}^*(Q_2) \cap \mathrm{cl}^*(Q_3)$, $\mathrm{cl}^*(Q_4) \cap \mathrm{cl}^*(Q_5)$, *or* $\mathrm{cl}^*(Q_6) \cap \mathrm{cl}^*(Q_1)$.

(ii) *If* $\mathbf{Q}$ *is spike-like, then there is at most one non-trivial series class in* $M \backslash a, b, c$*. This non-trivial series class has size at most* 3 *and the element of* $E(N)$ *corresponding to it is the unique element that is in* $\mathrm{cl}^*(Q_i)$ *for all* i *in* $\{1, 2, \ldots, 6\}$.

(iii) *If* $\mathbf{Q}$ *is a copaddle, then all non-trivial series classes have size at most* 3*. Elements of* $E(N)$ *corresponding to such series classes are in* $\mathrm{cl}^*(Q_i)$ *for all* i *in* $\{1, 2, \ldots, 6\}$.

From the last result, the reader may be tempted to think that, up to equivalence, all 3-separations exposed by a, b, or c can be seen from the flower $\mathbf{Q}$. The diagram in Figure 4 indicates that this is not the case.

Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Evidently $M \backslash a$ is or is not 3-connected. In the former case, by Bixby's Lemma [1], $\mathrm{co}(M \backslash a, b)$ or $\mathrm{si}(M \backslash a / b)$ is 3-connected. The final result of this section indicates precisely how to distinguish the different types of wild triangles.

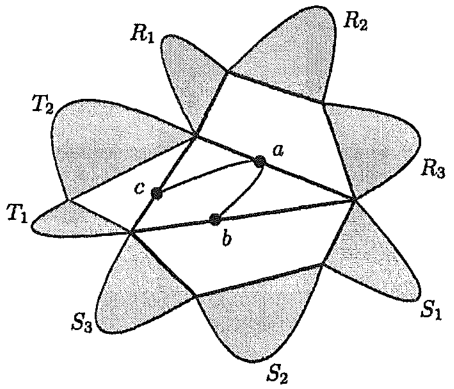

FIGURE 4. Inequivalent 3-separations are exposed by each of a and b.

Corollary 3.3. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Then $\{a, b, c\}$ is an internal triangle of a fan if and only if $M\backslash a$ is not 3-connected. Moreover, when $M\backslash a$ is 3-connected,*

(i) *$\{a, b, c\}$ is in a trident if and only if both $\mathrm{co}(M\backslash a, b)$ and $\mathrm{si}(M\backslash a/b)$ are 3-connected;*
(ii) *$\{a, b, c\}$ is a standard wild triangle if and only if $\mathrm{co}(M\backslash a, b)$ is 3-connected but $\mathrm{si}(M\backslash a/b)$ is not; and*
(iii) *$\{a, b, c\}$ is a costandard wild triangle if and only if $\mathrm{si}(M\backslash a/b)$ is 3-connected but $\mathrm{co}(M\backslash a, b)$ is not.*

4. AN EXTENSION OF TUTTE'S TRIANGLE LEMMA

The main theorem of the paper notes that one way in which a wild triangle can occur in a 3-connected matroid is as an internal triangle of a fan. In this section, we identify precisely when such wild triangles arise.

We begin the section by stating Tutte's Triangle Lemma [10]. This result is a useful tool in matroid structure theory and is used, for example, in the proofs of Tutte's Wheels-and-Whirls Theorem [10] and Seymour's Splitter Theorem [9].

Lemma 4.1. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. Suppose that $M\backslash b$ is not 3-connected, that no fan of M has b as an internal element, and that $|E(M)| \geq 4$. Then both $M\backslash a$ and $M\backslash c$ are 3-connected.*

The next theorem, the main result of this section, is an obvious strengthening of the last lemma. As such, it is of independent interest. Moreover, it has, as a straightforward consequence, Corollary 4.3, which splits wild triangles into two types and completely describes the first type.

Theorem 4.2. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. Suppose that $M\backslash b$ is not 3-connected, that no fan of M has b as an internal element, and that $|E(M)| \geq 4$. Then both $M\backslash a$ and $M\backslash c$ are 3-connected and neither a nor c exposes a 3-separation in M.*

Proof. By Tutte's Triangle Lemma, both $M\backslash a$ and $M\backslash c$ are 3-connected. Let (A, C) be a 2-separation of $M\backslash b$. As M is 3-connected, $\{a, c\}$ is not contained in A or C, so we may assume that $a \in A$ and $c \in C$.

First observe that

4.2.1. $|A|, |C| > 2$.

If $|A| = 2$, then A is a series pair in $M\backslash b$ so $A \cup b$ is a triad of M. It follows that $\{a, b, c\}$ is contained in a fan of M with at least four elements; a contradiction. Hence $|A| > 2$, and (4.2.1) follows by symmetry.

We show next that

4.2.2. $a \in \mathrm{cl}(A - a)$ *and* $c \in \mathrm{cl}(C - c)$.

If $a \notin \mathrm{cl}(A - a)$, then $(A - a, C \cup a)$ is a 2-separation of $M\backslash b$ with $\{a, c\} \subseteq C \cup a$. This contradiction implies that $a \in \mathrm{cl}(A - a)$, and (4.2.2) follows by symmetry.

4.2.3. $\lambda_{M\backslash a}(A - a) = 2$ *and* $\lambda_{M\backslash a}(C) = 2$.

We have $\lambda_{M\backslash b}(A) = 1$ and $a \in \mathrm{cl}(A - a)$, so $\lambda_{M\backslash b,a}(A - a) = 1$. Hence $\lambda_{M\backslash a}(A - a) \leq 2$. But $|A - a| \geq 2$ and $M\backslash a$ is 3-connected, so $\lambda_{M\backslash a}(A - a) = 2$. A similar, but easier, argument gives that $\lambda_{M\backslash a}(C) = 2$, so (4.2.3) holds.

Now assume that a exposes a 3-separation (R, G). Then, without loss of generality, $b \in G$ and $c \in R$. Next we show that

4.2.4. $(A - a) \cap G \neq \emptyset \neq (A - a) \cap R$.

If $(A - a) \cap G = \emptyset$, then $A - a \subseteq R$ so, by (4.2.2), $a \in \mathrm{cl}(R)$ contradicting the fact that a exposes (R, G). By symmetry, we conclude that (4.2.4) holds.

4.2.5. $(C - c) \cap R \neq \emptyset$.

If not, then $C - c \subseteq G$ so, by (4.2.2), $c \in \mathrm{cl}(G)$. Also $b \in G$. Hence $a \in \mathrm{cl}(G)$; a contradiction. Thus (4.2.5) holds.

4.2.6. $\lambda_{M\backslash a}(G \cap A) \leq 2$.

To see this, note that $\lambda_{M\backslash a}(G) = 2$ and $\lambda_{M\backslash a}(A - a) = 2$. Moreover, $|E - (G \cup (A - a))| \geq 2$ so, by uncrossing, $\lambda_{M\backslash a}(G \cap (A - a)) \leq 2$, that is, $\lambda_{M\backslash a}(G \cap A) \leq 2$.

4.2.7. $\lambda_{M\backslash a}(C \cap R) = 2$.

Since $\lambda_{M\backslash a}(R) = 2 = \lambda_{M\backslash a}(C)$, and $|(E - a) - (C \cup R)| \geq 2$ so, by uncrossing, $\lambda_{M\backslash a}(C \cap R) \leq 2$. But $|C \cap R| \geq 2$, so $\lambda_{M\backslash a}(C \cap R) = 2$.

4.2.8. $C \cap G \neq \emptyset$.

Assume $C \subseteq R$. Since $b \notin \mathrm{cl}(A)$, we have $b \notin \mathrm{cl}_{M\backslash a}(A-a)$, so $b \in \mathrm{cl}^*{}_{M\backslash a}(C)$. Hence $b \in \mathrm{cl}^*{}_{M\backslash a}(R)$. Thus, in $M\backslash a$, we have $(R,G) \cong (R \cup b, G-b)$. But $\{b,c\} \subseteq R \cup b$, so $a \in \mathrm{cl}(R \cup b)$. Hence a does not expose (R,G). This contradiction establishes (4.2.8).

4.2.9. $|C \cap G| \geq 2$.

Assume that $C \cap G = \{g\}$. If $C-g$ spans g, then $(R,G) \cong (R \cup g, G-g)$. By Lemma 2.4, $|G-g| \geq 3$. Hence $(R \cup g, G-g)$ is a 3-separation of $M\backslash a$ that is exposed by a. Replacing (R,G) by $(R \cup g, G-g)$, we get a contradiction to (4.2.8). Hence we may assume that $C-g$ does not span g. Thus $g \in \mathrm{cl}^*(A \cup b)$. But $b \in \mathrm{cl}^*(A)$, so $g \in \mathrm{cl}^*(A)$. Hence $g \in \mathrm{cl}^*{}_{M\backslash b}(A)$ and so $(A \cup g, C-g)$ is a 2-separation of $M\backslash b$ with $a \in A \cup g$ and $c \in C-g$. If we replace (A,C) by $(A \cup g, C-g)$, then we get a contradiction to (4.2.8). We deduce that (4.2.9) holds.

4.2.10. $\lambda_{M\backslash a}(A \cap R) \leq 2$.

We have $\lambda_{M\backslash a}(A-a) = 2 = \lambda_{M\backslash a}(R)$ and $|(E-a)-(A \cap R)| \geq 2$ by (4.2.9) so, by uncrossing, $\lambda_{M\backslash a}((A-a) \cap R) \leq 2$, that is, $\lambda_{M\backslash a}(A \cap R) \leq 2$.

4.2.11. $|A \cap R| > 1$ *or* $|A \cap G| > 1$.

Suppose that $|A \cap R| = 1 = |A \cap G|$. Then $|A| = 3$. Let $A \cap R = \{x\}$. Then $r(A) + r(C) = r(M\backslash b) + 1 = r(M) + 1$. But $a \in \mathrm{cl}(A-a)$ and so $r(A-a) + r(C) = r(M\backslash a) + 1$. Hence, as $M\backslash a$ is 3-connected, $r((A-a) \cup b) + r(C) = r(M\backslash a) + 2$, so $(A-a) \cup b$ is an independent triad of $M\backslash a$. Thus $x \in \mathrm{cl}^*{}_{M\backslash a}(((A-a) \cup b) - x)$. But $((A-a) \cup b) - x \subseteq G$, so $(R,G) \cong (R-x, G \cup x)$. As $(R-x, G \cup x)$ is a 3-separation of $M\backslash a$ exposed by a, we can replace (R,G) by $(R-x, G \cup x)$ to get a contradiction to (4.2.4). Thus (4.2.11) holds.

4.2.12. $|A \cap G| \neq 1 \neq |A \cap R|$.

Let $\{X,Y\} = \{R,G\}$ and assume that $A \cap X = \{x\}$. Then, by (4.2.11), $|A \cap Y| \geq 2$. If $x \in \mathrm{cl}(A \cap Y)$, then, as $a \in \mathrm{cl}(A-a)$, we deduce that $a \in \mathrm{cl}(Y)$, a contradiction. Thus $x \notin \mathrm{cl}(A \cap Y)$.

Now $A-a$ and Y are 3-separating in $M\backslash a$ and, since $|C \cap R|, |C \cap G| \geq 2$, the union $(A-a) \cup Y$ avoids at least two elements of $M\backslash a$. Hence, by uncrossing, $(A-a) \cap Y$, which equals $A \cap Y$, is 3-separating in $M\backslash a$. As $(A \cap Y) \cup x = A-a$, a 3-separating set in $M\backslash a$, we deduce that $x \in \mathrm{cl}_{M\backslash a}(A \cap Y)$ or $x \in \mathrm{cl}^*{}_{M\backslash a}(A \cap Y)$. The first possibility was eliminated above. Thus $x \in \mathrm{cl}^*{}_{M\backslash a}(A \cap Y) \subseteq \mathrm{cl}^*{}_{M\backslash a}(Y)$. Hence $(X,Y) \cong (X-x, Y \cup x)$. Replacing (X,Y) by $(X-x, Y \cup x)$, we get a contradiction to (4.2.4) since $(X-x) \cap (A-a) = \emptyset$. We conclude that (4.2.12) holds.

By (4.2.4) and (4.2.12), we have $|A \cap R| \geq 2$ and $|A \cap G| \geq 2$. Hence, by (4.2.6) and (4.2.10), $\lambda_{M\backslash a}(A \cap R) = 2 = \lambda_{M\backslash a}(A \cap G)$. Since $\lambda_{M\backslash a}(A-a) = 2$, we have $\lambda_{M\backslash a}(C \cup b) = 2$. Moreover, by uncrossing, $\lambda_{M\backslash a}((C \cap G) \cup b) = 2$. We deduce that $(A \cap R, C \cap R, (C \cap G) \cup b, A \cap G)$ is a flower Φ in $M\backslash a$.

Since $b \notin \text{cl}(C)$, we have $b \notin \text{cl}_{M\backslash a}(C)$, so $b \in \text{cl}^*{}_{M\backslash a}(A - a)$. Hence $b \in \text{cl}^*{}_{M\backslash a}((A \cap R) \cup (A \cap G)) - [(A \cap R) \cup (A \cap G)]$. Thus, by [7, Lemma 5.5(i)], $b \in \text{cl}^*{}_{M\backslash a}(A \cap G)$. Hence Φ is equivalent to $(A \cap R, C \cap R, C \cap G, (A \cap G) \cup b)$. Now $a \in \text{cl}(A - a)$ and so $c \in \text{cl}((A - a) \cup b)$, that is, $c \in \text{cl}((A \cap R) \cup (A \cap G) \cup b) - [(A \cap R) \cup (A \cap G) \cup b]$. Hence, by [7, Lemma 5.5(i)] again, $c \in \text{cl}(A \cap R)$. Thus $\{a, c\} \subseteq \text{cl}(A)$, so $b \in \text{cl}(A)$ and $(A \cup b, C)$ is a 2-separation of M; a contradiction. □

Corollary 4.3. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Then either*

(i) *none of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 3-connected, and M has a fan in which $\{a, b, c\}$ is an internal triangle; or*

(ii) *all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, and each of a, b, and c exposes a 3-separation of M.*

Proof. If all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, then (ii) holds. Hence we may assume that $M\backslash b$ is not 3-connected. Then, by Theorem 4.2, M has a fan having b as an internal element. Thus b is in a triad T^*. Now $M \not\cong U_{2,4}$ so, by orthogonality, we may assume that $T^* = \{b, c, d\}$ where $d \neq a$. Then $\{a, b, c, d\}$ is a fan of M. Hence $M\backslash c$ is not 3-connected. If $M\backslash a$ is not 3-connected, then, by Lemma 4.1, a is an internal element of a fan of M. Thus a is in a triad of M and (z, a, b, c, d) is a fan ordering of a fan in M. In this case, (i) holds.

We may now assume that $M\backslash a$ is 3-connected. We shall show that a does not expose a 3-separation of M. Suppose that M has a 3-separation (R, G) that is exposed by a. Then $|R|, |G| \geq 4$ and, by Lemma 2.8(i), we may assume that $b \in R$ and $c \in G$. Without loss of generality, $d \in R$. Then $R \supseteq \{b, d\}$ so $c \in \text{cl}^*{}_{M\backslash a}(R)$. Hence $(R \cup c, G - c)$ is an exactly 3-separating partition of $M\backslash a$ that is equivalent to (R, G). But $\{b, c\} \subseteq R \cup c$, so (R, G) is not well blocked by a; a contradiction. □

5. Towards the Main Result

The proof of the main result is long and essentially occupies the rest of the paper. In view of Corollary 4.3, we can make the following assumptions:

- M is a 3-connected matroid having $\{a, b, c\}$ as a triangle;
- all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected; and
- a, b, and c expose 3-separations in M.

These assumptions will remain in effect for the rest of the paper.

We shall take **A**, **B**, and **C** to be arbitrary 3-separations, $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2), in M exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$. The symmetries revealed here are summarized in Table 1. These symmetries will be constantly exploited. This section contains a number of basic observations about how the six sets A_1, A_2, B_1, B_2, C_1, and C_2 interact. This sets up the following section, which contains an overview of the logic of the proof of the main result.

a	b	c
B_2	C_2	A_2
C_1	A_1	B_1

TABLE 1. Location of the elements of $\{a, b, c\}$.

By Lemma 2.4, we have

5.0.1. $|A_1|, |A_2|, |B_1|, |B_2|, |C_1|, |C_2| \geq 4$.

Next we show that

5.0.2. $a \in \mathrm{cl}(B_2 - a) \cap \mathrm{cl}(C_1 - a), b \in \mathrm{cl}(C_2 - b) \cap \mathrm{cl}(A_1 - b), c \in \mathrm{cl}(A_2 - c) \cap \mathrm{cl}(B_1 - c)$.

By symmetry, it suffices to prove that $a \in \mathrm{cl}(B_2 - a)$. Assume not. Then $a \notin \mathrm{cl}_{M\backslash b}(B_2 - a)$ so, by duality, $a \in \mathrm{cl}^*{}_{M\backslash b}(B_1)$. Hence $(B_1 \cup a, B_2 - a) \cong (B_1, B_2)$. But $\{a, c\} \subseteq B_1 \cup a$, so (B_1, B_2) is not well blocked by b; a contradiction. Thus (5.0.2) holds.

5.0.3. $A_1 \cap B_1 \neq \emptyset$.

Since $b \in \mathrm{cl}(A_1 - b)$, if $A_1 \cap B_1 = \emptyset$, we obtain the contradiction that $b \in \mathrm{cl}(B_2)$.

5.0.4. $|A_1 \cap B_1| \geq 2$.

Suppose $A_1 \cap B_1 = \{x\}$. If $x \in \mathrm{cl}(A_2 \cap B_1)$, then $x \in \mathrm{cl}(A_2)$. Hence $(A_1, A_2) \cong (A_1 - x, A_2 \cup x)$ and replacing (A_1, A_2) by $(A_1 - x, A_2 \cup x)$ gives a contradiction to (5.0.3). We conclude that $x \notin \mathrm{cl}(A_2 \cap B_1)$, that is, $x \notin \mathrm{cl}(B_1 - x)$. It follows that $(B_1 - x, B_2 \cup x)$ is a 3-separation of $M\backslash b$ that is equivalent to (B_1, B_2). Replacing (B_1, B_2) by $(B_1 - x, B_2 \cup x)$, we again get a contradiction to (5.0.3). Thus (5.0.4) holds.

By symmetry, we deduce

5.0.5. $|A_i \cap B_i|, |B_i \cap C_i|, |C_i \cap A_i| \geq 2$ *for each* i *in* $\{1, 2\}$.

5.0.6. *If* $X \subseteq \{a, b, c\}$, *then*

$$\lambda_{M\backslash X}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1) = 2.$$

Since $r(M) = r(M\backslash a, b, c)$, it suffices to show that $\lambda_{M\backslash a,b,c}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1) = 2$. By (5.0.2), we have $c \in \mathrm{cl}(A_2 - c)$ and $a \in \mathrm{cl}(B_2 - a)$. Moreover, $\{a, c\}$ spans b. Thus $(A_2 \cup B_2) - \{a, c\}$ spans $\{a, b, c\}$ and so $\lambda_{M\backslash a,b,c}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1)$. Hence $\lambda_{M\backslash X}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1)$.

As $a \in \mathrm{cl}(B_2 - a)$, we have $\lambda_{M\backslash a,b}(B_2 - a) = \lambda_{M\backslash b}(B_2) = 2$. By symmetry, $\lambda_{M\backslash a,b}(A_1 - b) = 2$. But $A_2 = E - \{a, b\} - (A_1 - b)$, so $\lambda_{M\backslash a,b}(A_2) = 2$. Now, from the last paragraph, $\lambda_{M\backslash a,b}(A_1 \cap B_1) = \lambda_{M\backslash b}(A_1 \cap B_1)$. Since $M\backslash b$ is 3-connected, it follows by (5.0.5) that $\lambda_{M\backslash a,b}(A_1 \cap B_1) \geq 2$. By symmetry, $\lambda_{M\backslash a}(A_2 \cap B_2) = \lambda_{M\backslash a,b}(A_2 \cap B_2) \geq 2$. Thus

$$\begin{aligned} 2+2 &\leq \lambda_{M\backslash a,b}(A_1 \cap B_1) + \lambda_{M\backslash a,b}(A_2 \cap B_2) \\ &= \lambda_{M\backslash a,b}(A_1 \cap B_1) + \lambda_{M\backslash a,b}(E - \{a,b\} - (A_2 \cap B_2)) \\ &= \lambda_{M\backslash a,b}((A_1 - b) \cap B_1) + \lambda_{M\backslash a,b}((A_1 - b) \cup B_1) \\ &\leq \lambda_{M\backslash a,b}(A_1 - b) + \lambda_{M\backslash a,b}(B_1) \\ &= 2+2. \end{aligned}$$

Thus equality must hold throughout. Hence $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2$ and so (5.0.6) holds.

By symmetry with (5.0.6), we have the following.

5.0.7. *If $X \subseteq \{a,b,c\}$, then*

$$\begin{aligned} \lambda_{M\backslash X}(B_1 \cap C_1) &= \lambda_M(B_1 \cap C_1) = 2, \\ \lambda_{M\backslash X}(C_1 \cap A_1) &= \lambda_M(C_1 \cap A_1) = 2, \\ \lambda_{M\backslash X}(A_2 \cap B_2) &= \lambda_M(A_2 \cap B_2) = 2, \\ \lambda_{M\backslash X}(B_2 \cap C_2) &= \lambda_M(B_2 \cap C_2) = 2, \\ \lambda_{M\backslash X}(C_2 \cap A_2) &= \lambda_M(C_2 \cap A_2) = 2. \end{aligned}$$

5.0.8. $\lambda_M(A_2 \cap B_1) = \lambda_{M\backslash a}(A_2 \cap B_1) = \lambda_{M\backslash b}(A_2 \cap B_1) = \lambda_{M\backslash a,b}(A_2 \cap B_1)$;
$\lambda_M(B_2 \cap C_1) = \lambda_{M\backslash b}(B_2 \cap C_1) = \lambda_{M\backslash c}(B_2 \cap C_1) = \lambda_{M\backslash b,c}(B_2 \cap C_1)$;
$\lambda_M(C_2 \cap A_1) = \lambda_{M\backslash c}(C_2 \cap A_1) = \lambda_{M\backslash a}(C_2 \cap A_1) = \lambda_{M\backslash c,a}(C_2 \cap A_1)$.

Since $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, the first line holds; the second and third lines hold by symmetry.

5.0.9. $|(A_2 \cap B_1) - c| \geq 1, |(B_2 \cap C_1) - a| \geq 1$, *and* $|(C_2 \cap A_1) - b| \geq 1$.

Suppose that $A_2 \cap B_1 = \{c\}$. Then, since $\lambda_{M\backslash a}(A_2 \cap B_2) = 2 = \lambda_{M\backslash a}(A_2)$, we have $(A_1, A_2) \cong (A_1 \cup c, A_2 - c)$ in $M\backslash a$. But $\{b, c\} \subseteq A_1 \cup c$, so (A_1, A_2) is not exposed by a; a contradiction. We conclude that (5.0.9) holds.

5.0.10. *None of $A_1 \cap B_2, B_1 \cap C_2$, and $C_1 \cap A_2$ is empty.*

Suppose $A_1 \cap B_2 = \emptyset$. As $b \in \mathrm{cl}(A_1 - b) = \mathrm{cl}(A_1 \cap B_1)$, we deduce that $b \in \mathrm{cl}(B_1)$. This contradiction establishes that (5.0.9) holds.

5.0.11. $\lambda_M(A_2 \cap B_1) \in \{2,3\}$ *and* $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1,2\}$; *if* $\lambda_M(A_2 \cap B_1) = 3$, *then* $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$.

Since $b \in \mathrm{cl}(A_1 - b)$, we deduce that $\lambda_{M\backslash a,b}(A_2) = \lambda_{M\backslash a}(A_2) = 2$. By symmetry, $\lambda_{M\backslash a,b}(B_1) = 2$. Thus $\lambda_{M\backslash a,b}(A_2 \cap B_1) + \lambda_{M\backslash a,b}(A_2 \cup B_1) \leq 4$. Since $|(E - \{a,b\} - (A_2 \cup B_1))| = |A_1 \cap B_2| \geq 1$, we have $\lambda_{M\backslash a,b}(A_2 \cup B_1) \geq 1$, so $\lambda_{M\backslash a,b}(A_2 \cap B_1) \leq 3$. But $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, so $\lambda_M(A_2 \cap B_1) = \lambda_{M\backslash a,b}(A_2 \cap B_1) \leq 3$. By (5.0.9), $\lambda_M(A_2 \cap B_1) \geq 2$. Hence $\lambda_M(A_2 \cap B_1) \in \{2,3\}$. Since $\lambda_{M\backslash a,b}(A_2 \cup B_1) = \lambda_{M\backslash a,b}(A_1 \cap B_2)$, we deduce that $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1,2\}$. Moreover, if $\lambda_M(A_2 \cap B_1) = 3$, then $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. We conclude that (5.0.11) holds.

6. Overview

This section gives an overview of the logic of the argument to follow. The division of cases is based on the cardinality and connectivity of the sets $A_1 \cap B_2$ and $A_2 \cap B_1$. By (5.0.10) and (5.0.9), we know that $|A_1 \cap B_2| \geq 1$ and $|A_2 \cap B_1| \geq 2$. Moreover, by (5.0.11), $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1,2\}$. The argument will distinguish the following six cases:

(A) $|A_1 \cap B_2| = 1$;
(B) $|A_2 \cap B_1| = 2$;
(C) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| = 3$;
(D) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| = 2$;
(E) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| \geq 4$; and
(F) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| \geq 3$.

In case (A), Lemma 7.3 identifies three types of special structures that can arise after possibly replacing $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) by equivalent 3-separations. We call these structures pretridents of type I, II, and III. From a pretrident of type I, we immediately obtain a trident in M. In case (B), we show in Lemma 7.4 that $|B_1 \cap C_2| = 1$ or $|C_1 \cap A_2| = 1$ so, by symmetry, we have reduced to case (A) and again we find that $\{a, b, c\}$ is in a pretrident. In case (C), we show, in Lemma 7.5, that either $|E(M)| = 11$, or we can reduce to case (B) and hence to case (A). In case (D), Lemma 7.7 shows that $|E(M)| = 11$ or we can reduce to an earlier case. In case (E), which we shall treat last, we show that either a symmetric case to case (C) occurs, or the two sets of symmetric conditions to (E) also hold and outcome (ii) of Theorem 3.1 arises. Finally, in case (F), we show, in Lemma 8.4, that case (E) and its symmetric counterparts hold in the matroid M' that is obtained from M by performing a $\Delta - Y$ exchange in M on the triangle $\{a, b, c\}$ and then taking the dual of the result. Thus outcome (iii) of Theorem 3.1 arises.

Pretridents of type II and III do not appear in either of Theorems 1.1 or 3.1. In the next section, we initially describe the structure of M around $\{a, b, c\}$ relative to the 3-separations with which we begin, only allowing ourselves to replace these 3-separations by equivalent ones. Then, in Lemma 7.9, we show that, when $\{a, b, c\}$ is in a pretrident of type II or III, we can find a pretrident of type I containing $\{a, b, c\}$ by altering the choice of 3-separations exposed by a, b, and c to ones that need not be equivalent to those with which we began.

7. Tridents

In this section, we begin the treatment of the six cases noted in the preceding section. Specifically, we deal with cases (A)-(D) here. We begin with an elementary lemma. Recall that the assumptions noted at the outset of Section 5 are still in effect and that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are arbitrary 3-separations,

$(A_1, A_2), (B_1, B_2)$, and (C_1, C_2), exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$.

Lemma 7.1. *If* $|A_2 \cap B_1 \cap C_2| \geq 2$, *then*

$$\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2).$$

Proof. By (5.0.7), $2 = \lambda_{M\backslash b}(A_2 \cap C_2)$. Since $\lambda_{M\backslash b}(B_1) = 2$, we have, by uncrossing, that $\lambda_{M\backslash b}(A_2 \cap B_1 \cap C_2) = 2$. Hence $\lambda_M(A_2 \cap B_1 \cap C_2) = 2$ as $b \in \mathrm{cl}(A_1 - b)$. Since $a \in B_2$, we have $c \in \mathrm{cl}((A_1 - b) \cup B_2)$. Hence $\lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2) = 2$. □

The next lemma begins the treatment of case (A).

Lemma 7.2. *Suppose that* $|A_1 \cap B_2| = 1$ *and* $A_1 \cap B_2 \subseteq C_1$. *Then*

$$|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1.$$

Proof. Let $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$. Since $|A_1 \cap C_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_1| \geq 1$. Suppose that $|A_1 \cap B_1 \cap C_1| \geq 2$. We know by (5.0.7) that $A_1 \cap C_1$ and $A_1 \cap B_1$ are 3-separating in M and hence in $M\backslash b$. Their intersection has at least two elements, so their union is 3-separating in $M\backslash b$. Hence $\lambda_{M\backslash b}((A_1 \cap B_1) \cup r_{12}) = 2$. Thus, by Lemma 2.5, $r_{12} \in \mathrm{cl}^{(*)}_{M\backslash b}(A_1 \cap B_1)$. Hence $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$ in $M\backslash b$. But $b \in \mathrm{cl}(A_1 - b)$ and $A_1 - b \subseteq B_1 \cup r_{12}$ so (B_1, B_2) is not exposed by b. We deduce that $|A_1 \cap B_1 \cap C_1| = 1$, say $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

Now $|A_1| \geq 4$, so $|A_1 \cap B_1 \cap C_2| \geq 1$. Suppose $|A_1 \cap B_1 \cap C_2| \geq 2$. By (5.0.7), $\lambda_{M\backslash c}(A_1 \cap B_1) = 2$ and so, by Lemma 2.5, $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1 \cap B_1 \cap C_2)$. Thus $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$ in $M\backslash c$ and $|A_1 \cap (C_1 - r_{11})| = 1$; a contradiction to (5.0.5). Hence $|A_1 \cap B_1 \cap C_2| = 1$, say $A_1 \cap B_1 \cap C_1 = \{g_{11}\}$.

Next we show that $|A_2 \cap B_1 \cap C_1| = 1$. We have $|B_1 \cap C_1| \geq 2$, so $|A_2 \cap B_1 \cap C_1| \geq 1$. Assume that $|A_2 \cap B_1 \cap C_1| \geq 2$. We have $\lambda_{M\backslash a}(A_2) = 2 = \lambda_{M\backslash a}(B_1 \cap C_1)$. Thus, by uncrossing, $\lambda_{M\backslash a}(A_2 \cap B_1 \cap C_1) = 2$. Since $\lambda_M((A_2 \cap B_1 \cap C_1) \cup r_{11}) = 2 = \lambda_{M\backslash a}((A_2 \cap B_1 \cap C_1) \cup r_{11})$, we have $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_2 \cap B_1 \cap C_1)$ so $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_2)$. Hence $(A_1, A_2) \cong (A_1 - r_{11}, A_2 \cup r_{11})$ in $M\backslash a$. But, replacing (A_1, A_2) by $(A_1 - r_{11}, A_2 \cup r_{11})$ gives a contradiction to (5.0.5) since $|(A_1 - r_{11}) \cap B_1| = 1$. We deduce that $|A_2 \cap B_1 \cap C_1| = 1$. □

Lemma 7.3. *Suppose that* $|A_1 \cap B_2| = 1$. *Then, after*

(i) *the possible replacement of* $(A_1, A_2), (B_1, B_2)$, *and* (C_1, C_2) *by equivalent 3-separations;*

(ii) *a possible relabelling of* $(A_1, A_2, a, B_1, B_2, b, C_1, C_2, c)$ *by* $(B_2, B_1, b, A_2, A_1, a, C_2, C_1, c)$; *and*

(iii) *a possible rotation of the labels on the triples* $(A_1, A_2, a), (B_1, B_2, b)$, *and* (C_1, C_2, c);

the following hold:

(a) $|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1$; *and*

(b) *A_1 is a quad of $M\backslash a$ and $A_1 \cup a$ is a cocircuit of M.*

In addition,

(I) *$|A_2 \cap B_1 \cap C_2| = 0$ and $A_2 \cap B_2 \subseteq C_2$; or*

(II) *$|A_2 \cap B_1 \cap C_2| = 0$ and $\lambda_M(A_2 \cap B_2 \cap C_2) = 2 = \lambda_{M\backslash a,c}(A_2 \cap C_1) = \lambda_M(A_2 \cap B_2 \cap C_1)$; or*

(III) *$\lambda_M(A_2 \cap B_2 \cap C_2) = 2 = \lambda_{M\backslash a,c}(A_2 \cap C_1) = \lambda_M(A_2 \cap B_2 \cap C_1)$ and $\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(B_1 \cap C_2)$.*

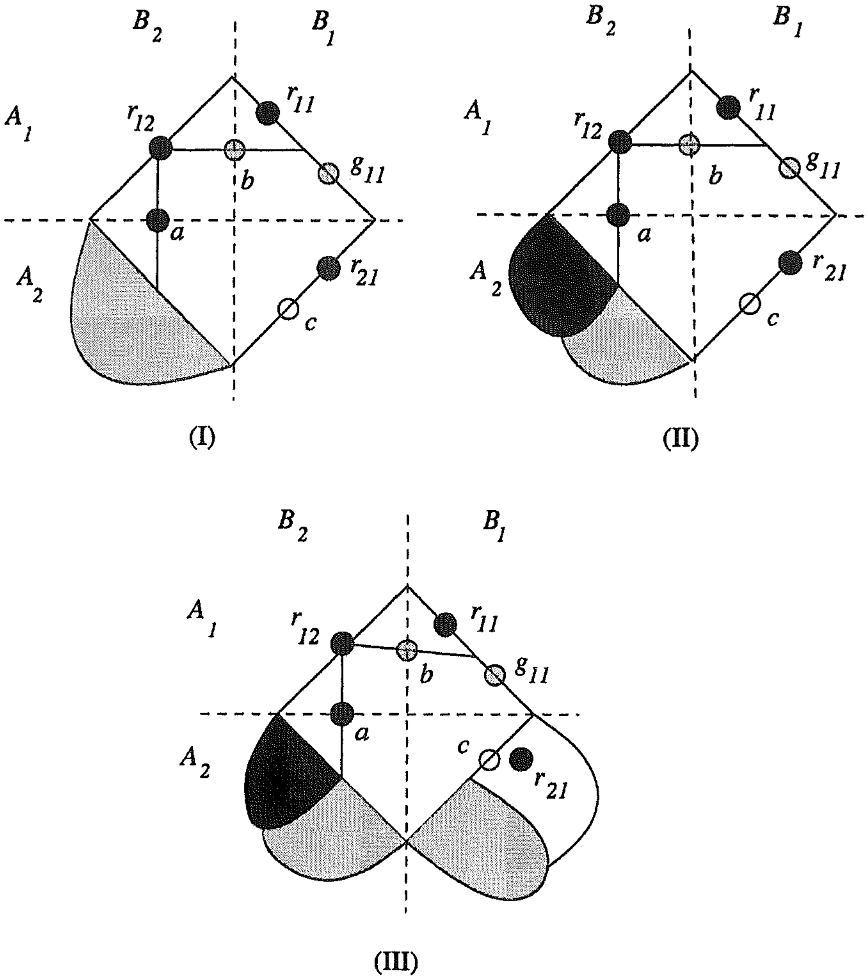

FIGURE 5. The three types of pretrident.

The situations corresponding to (I), (II), and (III) are illustrated in Figure 5. Each of the three parts of the diagram should be interpreted as basically a Venn diagram. The elements of C_1 correspond to black points while those in C_2 are shaded gray. Regions that are shaded indicate the presence of at least two elements. The placement of the elements a and b is to indicate that their deletion from M exposes the 3-separations (A_1, A_2) and (B_1, B_2), respectively.

To achieve outcomes (I)-(III) of Lemma 7.3, we allow equivalence moves and relabelling as described in (i)-(iii) of the lemma. When we can manipulate $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in this way so that (I), (II), or (III) in Figure 5 occurs, we shall say that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a *pretrident of type I, type II, or type III*, respectively, or that $\{a, b, c\}$ *occurs in a pretrident with respect to* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$. We observe that, when $\{a, b, c\}$ occurs in a pretrident of type I, each of A_1, B_1, and C_1 has exactly four elements and so, by Lemma 2.4, these sets are quads of $M\backslash a$, $M\backslash b$, and $M\backslash c$, respectively. Thus $A_1 \cup B_1 \cup C_1$ is a trident in M containing $\{a, b, c\}$. In fact, we shall show in Lemma 7.9 that, when $\{a, b, c\}$ occurs in any one of the three types of pretridents, $\{r_{11}, r_{12}, r_{21}, g_{11}, a, b, c\}$ is a trident in M, where elements are labelled as in Figure 5.

Proof of Lemma 7.3. Without loss of generality, we may assume that $A_1 \cap B_2 \subseteq C_1$. Thus $B_2 \cap C_2 = A_2 \cap B_2 \cap C_2$. Moreover, by Lemma 7.2,

$$|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1.$$

We denote the elements of these three sets by r_{11}, g_{11}, and r_{21}, respectively. Let $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$. Since $|A_1| = 4$, by Lemma 2.4, we must have that A_1 is a quad of $M\backslash a$. As $b \in A_1$, it follows by orthogonality with the triangle $\{a, b, c\}$ that $A_1 \cup a$ is a cocircuit of M.

Since $|B_2 \cap C_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_2| \geq 2$. Suppose that $|A_2 \cap B_2 \cap C_1| = 1$, say $A_2 \cap B_2 \cap C_1 = \{r_{22}\}$. Then $r_{22} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_2 \cap C_2)$. Thus, by replacing (C_1, C_2) by $(C_1 - r_{22}, C_2 \cup r_{22})$, an equivalent 3-separating partition of $M\backslash c$, we reduce to the case when $|A_2 \cap B_2 \cap C_1| = 0$. We deduce that we may assume that either

(i) $|A_2 \cap B_2 \cap C_1| = 0$; or
(ii) $|A_2 \cap B_2 \cap C_1| \geq 2$.

In case (ii), $\lambda_M(A_2 \cap B_2 \cap C_1) \geq 2$. Now, by (5.0.11), $\lambda_{M\backslash a,c}(A_2 \cap C_1) \in \{1, 2\}$. Moreover, $\lambda_M(A_2 \cap B_2 \cap C_1) = \lambda_{M\backslash a,c}(A_2 \cap B_2 \cap C_1)$. If $\lambda_{M\backslash a,c}(A_2 \cap B_2 \cap C_1) > \lambda_{M\backslash a,c}(C_1 \cap A_2)$, then, by Lemma 2.7, $r_{21} \in \mathrm{cl}(A_2 \cap B_2 \cap C_1)$. Thus we can replace (B_1, B_2) by the equivalent 3-separating partition $(B_1 - r_{21}, B_2 \cup r_{21})$ to get a contradiction to (5.0.5). We deduce that, in case (ii), $\lambda_M(A_2 \cap B_2 \cap C_1) = 2$ and $\lambda_{M\backslash a,c}(C_1 \cap A_2) = 2$. Hence our two cases become:

(i) $|A_2 \cap B_2 \cap C_1| = 0$; or
(ii) $|A_2 \cap B_2 \cap C_1| \geq 2$ and $\lambda_M(A_2 \cap B_2 \cap C_1) = 2 = \lambda_{M\backslash a,c}(C_1 \cap A_2)$.

Now consider $|A_2 \cap B_1|$. If this is 2, then case (I) or (II) holds depending on which of (i) and (ii) holds. If $|A_2 \cap B_1| = 3$, then $A_2 \cap B_1 \cap C_2 = \{g_{21}\}$, say. Since both $B_2 \cap C_2$ and $(B_2 \cap C_2) \cup g_{21} = A_2 \cap C_2$ are exactly 3-separating set in $M\backslash b$, we deduce that $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash b}(B_2 \cap C_2)$. Thus, after replacing (B_1, B_2) by the equivalent 3-separation $(B_1 - g_{21}, B_2 \cup g_{21})$, we have reduced to the case when $|A_2 \cap B_1| = 2$. Again case (I) or (II) holds.

We may now assume that $|A_2 \cap B_1| \geq 4$, so $|A_2 \cap B_1 \cap C_2| \geq 2$. Then, by a symmetric argument to that given in the penultimate paragraph, we deduce that $\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(B_1 \cap C_2)$. If case (i) occurs, then $|C_1 \cap A_2| = 1$. By rotating the labels on the triples $(A_1, A_2, a), (B_1, B_2, b)$, and (C_1, C_2, c), we obtain that case (I) or case (II) holds. If case (ii) occurs, then we have that case (III) holds. □

Next we show that if case (B) arises, then, after a symmetric relabelling, we obtain case (A).

Lemma 7.4. *If $|A_2 \cap B_1| = 2$, then either*

(i) $|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = |B_1 \cap C_2| = 1$ *and* $|A_2 \cap B_1 \cap C_2| = 0$; *or*

(ii) $|A_2 \cap B_2 \cap C_1| = |A_2 \cap B_2 \cap C_2| = |A_2 \cap B_1 \cap C_2| = |C_1 \cap A_2| = 1$ *and* $|A_2 \cap B_1 \cap C_1| = 0$.

In each case, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident.

Proof. Suppose $(A_2 \cap B_1) - c \subseteq C_1$, say $(A_2 \cap B_1) - c = \{r_{21}\}$. Since $|B_1 \cap C_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_1| \geq 1$. If $|A_1 \cap B_1 \cap C_1| \geq 2$, then, since $B_1 \cap C_1$ and $A_1 \cap B_1$ are exactly 3-separating in $M\backslash a$, so too is their union. Hence $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_1 \cap B_1)$, so $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_1)$. Replacing (A_1, A_2) by the equivalent 3-separating partition $(A_1 \cup r_{21}, A_2 - r_{21})$ gives a contradiction to (5.0.9). Hence $|A_1 \cap B_1 \cap C_1| = 1$, say $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

If $|A_1 \cap B_1 \cap C_2| \geq 2$, then, as $A_1 \cap B_1$ is exactly 3-separating in $M\backslash c$, it follows that $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1 \cap B_1 \cap C_2)$. Thus $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$ in $M\backslash c$. But $|B_1 \cap (C_1 - r_{11})| = 1$ so we have contradicted (5.0.5). Hence $|A_1 \cap B_1 \cap C_2| = 1$. Thus $|B_1 \cap C_2| = 1$. Hence if $(A_2 \cap B_1) - c \subseteq C_1$, then (i) holds. By symmetry, if $(A_2 \cap B_1) - c \subseteq C_2$, then (ii) holds.

In cases (i) and (ii), we have $|B_1 \cap C_2| = 1$ and $|C_1 \cap A_2| = 1$, respectively. These are symmetric to the case $|A_1 \cap B_2| = 1$ so, by Lemma 7.3, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident. □

The next lemma treats case (C).

Lemma 7.5. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| = 3$, then*

(i) *$A_2 \cap B_1$ is a triangle; and*

(ii) *either $|E(M)| = 11$, or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident.*

Proof. By (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. Since $A_2 \cap B_1$ has three elements, it is a triangle or a triad of M. The triangle $\{a, b, c\}$ implies that $A_2 \cap B_1$ is not a triad, so it is a triangle. Since $c \notin \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)$, we deduce that $|A_2 \cap B_1 \cap C_1| = 1 = |A_2 \cap B_1 \cap C_2|$. Let $A_2 \cap B_1 \cap C_2 = \{g_{21}\}$. Since $|A_2 \cap C_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_2| \geq 1$. Assume $|A_2 \cap B_2 \cap C_2| \geq 2$. Then, as $\lambda_M(A_2 \cap B_2) = 2$ and $\lambda_M(A_2 \cap C_2) = 2$, it follows by uncrossing that $\lambda_M((A_2 \cap B_2) \cup g_{21}) = 2$. Hence $g_{21} \in \mathrm{cl}_M^{(*)}(A_2 \cap B_2)$. As $A_2 \cap B_1$ is a triangle, it follows that $g_{21} \in \mathrm{cl}(A_2 \cap B_2) \subseteq \mathrm{cl}(B_2)$. Thus $(B_1, B_2) \cong (B_1 - g_{21}, B_2 \cup g_{21})$ in $M\backslash c$. Replacing (B_1, B_2) by $(B_1 - g_{21}, B_2 \cup g_{21})$, we get that $|A_2 \cap (B_1 - g_{21})| = 2$. Hence, by Lemma 7.4, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident. We may now assume that $|A_2 \cap B_2 \cap C_2| = 1$, say $A_2 \cap B_2 \cap C_2 = \{g_{22}\}$.

Since $|A_2 \cap B_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_1| \geq 1$. Assume $|A_2 \cap B_2 \cap C_1| \geq 2$. Now $\lambda_{M\backslash c}(A_2 \cap B_2) = 2 = \lambda_{M\backslash c}(C_1)$. Thus, by uncrossing, $\lambda_{M\backslash c}(A_2 \cap B_2 \cap C_1) = 2$. Hence $g_{22} \in \mathrm{cl}_{M\backslash c}^{(*)}(A_2 \cap B_2 \cap C_1) \subseteq \mathrm{cl}_{M\backslash c}^{(*)}(C_1)$. Hence $(C_1, C_2) \cong (C_1 \cup g_{22}, C_2 - g_{22})$ in $M\backslash c$. But $|(C_2 - g_{22}) \cap A_2| = 1$; a contradiction to (5.0.5). Hence $|A_2 \cap B_2 \cap C_1| = 1$.

By the symmetry between $A_2 \cap B_2$ and $A_1 \cap B_1$, we deduce that $|A_1 \cap B_1 \cap C_1| = 1 = |A_1 \cap B_1 \cap C_2|$. Now $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$. Thus, either $|A_1 \cap B_2| = 2$ and so $|E(M)| = 11$, or $|A_1 \cap B_2| > 2$. Since we have assumed that $|E(M)| \neq 11$, we deduce that $|A_1 \cap B_2| > 2$. Thus, without loss of generality, $|A_1 \cap B_2 \cap C_2| \geq 2$. Thus $\lambda_M(A_1 \cap B_2 \cap C_2) \geq 2 = \lambda_M((A_1 \cap B_2 \cap C_2) \cup g_{22})$. Hence, by Lemma 2.5 or 2.7, we have $g_{22} \in \mathrm{cl}^{(*)}(A_1 \cap B_2 \cap C_2)$, so $g_{22} \in \mathrm{cl}_{M\backslash a}^{(*)}(A_1 \cap B_2 \cap C_2) \subseteq \mathrm{cl}_{M\backslash a}^{(*)}(A_1)$. Hence $(A_1, A_2) \cong (A_1 \cup g_{22}, A_2 - g_{22})$ in $M\backslash a$. But $|(A_2 - g_{22}) \cap B_2| = 1$, contradicting (5.0.5). $\square$

In combination with the last lemma, the next lemma guarantees that, when case (E) arises but none of cases (A)-(C) arise, we can assume that we have symmetry between (A_1, A_2), (B_1, B_2), and (C_1, C_2).

Lemma 7.6. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| > 3$, then $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 2$ and $\lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$.*

Proof. By symmetry, it suffices to prove the first equation. By (5.0.11), we may assume that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$. We have $2 = \lambda_{M\backslash b}(B_1)$ and $c \in \mathrm{cl}(B_1 - c)$, so $2 = \lambda_{M\backslash b}(B_1 - c) = \lambda_{M\backslash b,c}(B_1 - c)$. Since $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$, by (5.0.11) we have $\lambda_{M\backslash a,b}(A_2 \cap B_1) = 2$. Then, since $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, we have $\lambda_M(A_2 \cap B_1) = 2$.

Now $|A_2 \cap B_1| > 3$. Thus $c \in \mathrm{cl}^{(*)}((A_2 \cap B_1) - c)$. But $\{a, b, c\}$ is a triangle, so $c \notin \mathrm{cl}^*((A_2 \cap B_1) - c)$. Hence $c \in \mathrm{cl}((A_2 \cap B_1) - c)$. Thus $\lambda_{M\backslash c}((A_2 \cap B_1) - c) = 2$, so $\lambda_{M\backslash b,c}((A_2 \cap B_1) - c) = 2$ since $b \in \mathrm{cl}(A_1 - b)$.

Since $c \in \mathrm{cl}((A_2 \cap B_1) - c)$ but $c \notin \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)$, we deduce that both $B_1 \cap C_2 \cap A_2$ and $B_1 \cap C_1 \cap A_2$ are nonempty. Since $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$ and $\lambda_{M\backslash b,c}((A_2 \cap B_1) - c) = 2$, we have, by Lemma 2.2, that $\lambda_{M\backslash b,c}(B_1 \cap C_2 \cap A_2) = 1$ or $\lambda_{M\backslash b,c}(((A_2 \cap B_1) - c) \cup (B_1 \cap C_2)) = 1$. But the latter implies that $\lambda_{M\backslash b}((A_2 \cap B_1) \cup (B_1 \cap C_2)) = 1$; a contradiction since $M\backslash b$ is 3-connected. Thus $\lambda_{M\backslash b,c}(B_1 \cap C_2 \cap A_2) = 1$ so, by Lemma 7.1, $|B_1 \cap C_2 \cap A_2| = 1$. Let $B_1 \cap C_2 \cap A_2 = \{g_{21}\}$.

Since $|A_2 \cap B_1| > 3$, we have $|A_2 \cap B_1 \cap C_1| \geq 2$. Since $\lambda_M(A_2 \cap B_1) = 2$ and $\lambda_M(B_1 \cap C_1) = 2$, it follows by uncrossing that $\lambda_M(A_2 \cap B_1 \cap C_1) = 2$ and hence $\lambda_{M\backslash c}(A_2 \cap B_1 \cap C_1) = 2$. Thus, by Lemma 2.5, $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_1 \cap C_1)$, so $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(C_1)$. Therefore $(C_1, C_2) \cong (C_1 \cup g_{21}, C_2 - g_{21})$ in $M\backslash c$. But $c \in \mathrm{cl}(C_1 \cup g_{21})$; a contradiction. □

The next lemma treats case (D).

Lemma 7.7. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| = 2$, then $|E(M)| = 11$ or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident.*

Proof. Assume that $|E(M)| \neq 11$ and that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident. Suppose that $A_1 \cap B_2 \subseteq C_1$. We have $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $b \in \mathrm{cl}(C_2 - b)$. Hence $\lambda_{M\backslash a}(A_1 \cap B_2) = 1$; a contradiction. Thus $|A_1 \cap B_2 \cap C_2| \geq 1$. By symmetry, $|A_1 \cap B_2 \cap C_1| \geq 1$. Hence $|A_1 \cap B_2 \cap C_2| = 1 = |A_1 \cap B_2 \cap C_1|$, say $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$ and $A_1 \cap B_2 \cap C_2 = \{g_{12}\}$.

If $|A_1 \cap B_1 \cap C_1| \geq 2$, then, as $A_1 \cap B_1$ and $B_1 \cap C_1$ are both exactly 3-separating in $M\backslash b$, so is their intersection. Since $(A_1 \cap B_1 \cap C_1) \cup r_{12}$ is also exactly 3-separating in $M\backslash b$, we deduce that $r_{12} \in \mathrm{cl}^{(*)}_{M\backslash b}(A_1 \cap B_1 \cap C_1)$. Hence $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$. But $|A_1 \cap (B_2 - r_{12})| = 1$ and so, by Lemma 7.3, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident; a contradiction. Hence we may assume that $|A_1 \cap B_1 \cap C_1| \leq 1$. Since $|A_1 \cap C_1| \geq 2$, we get $|A_1 \cap B_1 \cap C_1| = 1$. By symmetry, $|A_2 \cap B_2 \cap C_2| = 1$. Let $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

As $|A_1 \cap B_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_2| \geq 1$. If $|A_1 \cap B_1 \cap C_2| \geq 2$, then $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1 \cap B_1 \cap C_2)$ so $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$. Since $|A_1 \cap (C_1 - r_{11})| = 1$, we have a contradiction to (5.0.5). Hence $|A_1 \cap B_1 \cap C_2| = 1$ and, by symmetry, $|A_2 \cap B_2 \cap C_1| = 1$.

As $|A_2 \cap C_2|, |B_1 \cap C_1| \geq 2$, we deduce that $|A_2 \cap B_1 \cap C_2| \geq 1$ and $|A_2 \cap B_1 \cap C_1| \geq 1$. If equality holds in both of these, then $|E(M)| = 11$; a contradiction. By symmetry, we may assume that $|A_2 \cap B_1 \cap C_2| \geq 2$. Then, by Lemma 7.1, $\lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2) = 2$. Now $|B_2 \cap C_1| > 2$. Suppose that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 2$. If $|B_2 \cap C_1| = 3$, then, by Lemma 7.5, $|E(M)| = 11$ or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident; a contradiction. Hence $|B_2 \cap C_1| > 3$ and, by Lemma 7.6, $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$; a contradiction. We may now assume that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$. Then $\lambda_{M\backslash b,c}((A_2 \cap B_1 \cap C_2) \cup g_{11}) = 1$ where $A_1 \cap B_1 \cap C_2 = \{g_{11}\}$. By Lemma 2.7, $g_{11} \in \mathrm{cl}_{M\backslash b,c}(A_2 \cap B_1 \cap C_2)$. Hence $(A_1, A_2) \cong (A_1 - g_{11}, A_2 \cup g_{11})$. But $|(A_1 - g_{11}) \cap B_1| = 1$, a contradiction to (5.0.5). □

The next result summarizes the lemmas to date in this section. It notes that when any of cases (A)-(D) occurs, by replacing (A_1, A_2), (B_1, B_2), and (C_1, C_2) by equivalent 3-separations and performing a symmetric relabelling, we get one of the three outcomes shown in Figure 5.

Corollary 7.8. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M, where $|E(M)| \neq 11$. Suppose that all of $M\backslash a, M\backslash b$, and $M\backslash c$ are 3-connected and* *that* (A_1, A_2), (B_1, B_2), *and* (C_1, C_2) *are 3-separations exposed by* a, b, *and* c, *respectively, with* $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, *and* $c \in A_2 \cap B_1$. *If*

(A) $|A_1 \cap B_2| = 1$, *or*
(B) $|A_2 \cap B_1| = 2$, *or*
(C) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ *and* $|A_2 \cap B_1| = 3$, *or*
(D) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ *and* $|A_1 \cap B_2| = 2$,

then $\{a, b, c\}$ *is in a pretrident with respect to* (A_1, A_2), (B_1, B_2), *and* (C_1, C_2).

We conclude this section by showing that, when $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, we can choose potentially different 3-separations $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ exposed by b and c so that $(a, b, c, \mathbf{A}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is a pretrident of type I and hence $\{a, b, c\}$ is in a trident. It should be noted that when we choose $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ so that $\hat{B}_1$ and $\hat{C}_1$ are quads of $M\backslash b$ and $M\backslash c$, we have no guarantee that these new 3-separations are equivalent to those with which we began.

Lemma 7.9. *Suppose that* $|A_1 \cap B_2| = 1$ *and* $A_1 \cap B_2 \subseteq C_1$. *Then* A_1 *is a quad of* $M\backslash a$. *Moreover, there are 3-separations* $(\hat{B}_1, \hat{B}_2)$ *and* $(\hat{C}_1, \hat{C}_2)$ *that are exposed by* b *and* c, *respectively, such that* $\hat{B}_1$ *and* $\hat{C}_1$ *are quads in* $M\backslash b$ *and* $M\backslash c$. *In particular,* $A_1 \cup \hat{B}_1 \cup \hat{C}_1$ *is a trident in* M.

Proof. From Lemma 7.3, we know that A_1 is a quad of $M\backslash a$ and that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type I, II, or III. If this pretrident has type I, then the lemma holds by taking $(\hat{B}_1, \hat{B}_2) = (B_1, B_2)$, and $(\hat{C}_1, \hat{C}_2) = (C_1, C_2)$. We may now assume that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type II or III. We shall maintain the same labelling of elements as before (see Figure 5).

To complete the proof of the lemma, we shall show that

(i) $\{r_{11}, r_{12}, r_{21}, a\}$ is a quad of $M\backslash c$ and $(\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$ is exposed by c; and
(ii) $\{r_{11}, r_{21}, g_{11}, c\}$ is a quad of $M\backslash b$ and $(\{r_{11}, r_{21}, g_{11}, c\}, E - b - \{r_{11}, r_{21}, g_{11}, c\})$ is exposed by b.

First observe that $\lambda_{M\backslash c}((A_1 \cup B_1) - c) = 2 = \lambda_{M\backslash c}(C_1)$. Hence, by uncrossing, $\lambda_{M\backslash c}(C_1 \cap (A_1 \cup B_1)) = 2$, that is, $\{r_{11}, r_{12}, r_{21}, a\}$ is 3-separating in $M\backslash c$. We show first that $\{r_{11}, r_{12}, r_{21}, a\}$ is a circuit of M. Assume not. Then it contains a triangle. But $a \notin \mathrm{cl}(A_1)$, so $\{r_{11}, r_{12}, a\}$ is not a triangle. If $\{r_{11}, r_{21}, a\}$ is a triangle, then $\{a, c\} \subseteq \mathrm{cl}(B_1)$, so $b \in \mathrm{cl}(B_1)$; a contradiction. If $\{r_{12}, r_{21}, a\}$ is a triangle, then $(B_1, B_2) \cong (B_1 - r_{21}, B_2 \cup r_{21})$. But $|(B_1 - r_{21}) \cap C_1| = 1$; a contradiction to (5.0.5). Finally, if $\{r_{11}, r_{12}, r_{21}\}$ is a triangle, then $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$. But $A_1 \cap (B_2 - r_{12}) = \emptyset$; a contradiction to (5.0.10). We conclude that $\{r_{11}, r_{12}, r_{21}, a\}$ is a circuit of M.

Next we show that $\{r_{11}, r_{12}, r_{21}, a\}$ is a cocircuit of $M\backslash c$. Assume not. Then this set contains a triad of $M\backslash c$. If $\{r_{11}, r_{12}, r_{21}\}$ is a triad of $M\backslash c$, then, by orthogonality, it is a triad of M and hence of $M\backslash b$. Thus $(B_1, B_2) \cong$

$(B_1 \cup r_{12}, B_2 - r_{12})$. As $A_1 \cap (B_2 - r_{12}) = \emptyset$, we contradict (5.0.10). If $\{r_{11}, r_{12}, a\}$ is a triad of $M\backslash c$, then, by orthogonality, $\{r_{11}, r_{12}, a, c\}$ is a cocircuit of M, so $\{r_{11}, r_{12}, c\}$ is a triad of $M\backslash a$. Then $(A_1, A_2) \cong (A_1 \cup c, A_2 - c)$. But $a \in \mathrm{cl}(A_1 \cup c)$; a contradiction. If $\{r_{11}, r_{21}, a\}$ is a triad of $M\backslash c$, then $\{r_{11}, r_{21}, c\}$ is a triad of $M\backslash a$. Thus $(A_1, A_2) \cong (A_1 - r_{11}, A_2 \cup r_{11})$. But $|(A_1 - r_{11}) \cap B_1| = 1$, contradicting (5.0.5). Finally, if $\{r_{12}, r_{21}, a\}$ is a triad of $M\backslash c$, then $\{r_{12}, r_{21}, c\}$ is a triad of $M\backslash a$, so $(A_1, A_2) \cong (A_1 - r_{12}, A_2 \cup r_{12})$ and we get a contradiction since $(A_1 - r_{12}) \cap B_2 = \emptyset$. We conclude that $\{r_{11}, r_{12}, r_{21}, a\}$ is a cocircuit of $M\backslash c$. Hence $\{r_{11}, r_{12}, r_{21}, a, c\}$ is a cocircuit of M.

We show next that $(\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$ is exposed by c. First observe that $c \notin \mathrm{cl}(E - c - \{r_{11}, r_{12}, r_{21}, a\})$ since $\{r_{11}, r_{12}, r_{21}, a, c\}$ is a cocircuit of M. Because $E - c - \{r_{11}, r_{12}, r_{21}, a\}$ is fully closed in $M\backslash c$, we need only consider the full closure of $\{r_{11}, r_{12}, r_{21}, a\}$ in $M\backslash c$. Since $\{r_{11}, r_{12}, r_{21}, a\} \subseteq C_1$, it follows that $\mathrm{fcl}_{M\backslash c}(\{r_{11}, r_{12}, r_{21}, a\}) \subseteq \mathrm{fcl}_{M\backslash c}(C_1)$. Now $c \notin \mathrm{cl}(\mathrm{fcl}_{M\backslash c}(C_1))$ so $c \notin \mathrm{cl}(\mathrm{fcl}_{M\backslash c}(\{r_{11}, r_{12}, r_{21}, a\})$. We conclude that (i) holds. Let $\hat{\mathbf{C}} = (\hat{C}_1, \hat{C}_2) = (\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$.

If $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type II, then, since $\hat{\mathbf{C}}$ is exposed by c, we see that $(a, b, c, \mathbf{A}, \mathbf{B}, \hat{\mathbf{C}})$ is a pretrident of type I.

Finally, let $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ be a pretrident of type III. Then, by redrawing the figure, we see that $(c, a, b, \hat{\mathbf{C}}, \mathbf{A}, \mathbf{B})$ is a pretrident of type II. After this move, we see that $\{r_{11}, r_{21}, g_{11}, c\}$ is in a symmetric position to that of $\{r_{11}, r_{12}, r_{21}, a\}$ before the move. We deduce that (ii) holds. In particular, letting $\hat{\mathbf{B}} = (\hat{B}_1, \hat{B}_2) = (\{r_{11}, r_{21}, g_{11}, c\}, E - b - \{r_{11}, r_{21}, g_{11}, c\})$, we have that $\hat{\mathbf{B}}$ is a 3-separation exposed by b. Thus $(c, a, b, \hat{\mathbf{C}}, \mathbf{A}, \hat{\mathbf{B}})$ is a pretrident of type I. Redrawing again, we find that $(a, b, c, \mathbf{A}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is a pretrident of type I. □

8. A Delta-Wye Exchange

In this section, we show that if case (F) occurs in M, then, after performing a $\Delta - Y$ exchange and taking the dual of the result, we get a matroid in which case (E) and the two sets of symmetric conditions occur.

Lemma 8.1. *Assume that $|E(M)| \neq 11$, that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, and that $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. Then*

(i) $|A_2 \cap B_1| \geq 3$, $|B_2 \cap C_1| \geq 3$, $|C_2 \cap A_1| \geq 3$;
(ii) $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1 = \lambda_{M\backslash c,a}(C_1 \cap A_2)$; *and*
(iii) $|A_1 \cap B_2| \geq 3$, $|B_1 \cap C_2| \geq 3$, $|C_1 \cap A_2| \geq 3$.

Proof. Part (i) follows from Lemma 7.4; part (ii) follows using part (i), symmetry, and Lemmas 7.5 and 7.6; and part (iii) follows from (ii) by Lemma 7.7 and symmetry. □

The next two lemmas introduce the matroid M' that appears in outcome (iii) of Theorem 3.1 and then prove that M' satisfies the conditions imposed on M at the start of Section 5.

Lemma 8.2. *Let Δ be the triangle $\{a,b,c\}$ of M and consider a copy of $M(K_4)$ that has Δ as a triangle and has $\{a',b',c'\}$ as the complementary triad, where e' is the element of $M(K_4)$ that is not in a triangle with e. Let $\Delta M = P_\Delta(M(K_4),M)\backslash\Delta$, that is, ΔM is obtained from M by a $\Delta - Y$ exchange on Δ. Then*

(i) *ΔM is 3-connected;*
(ii) *for all $\{x,y,z\} = \{a,b,c\}$, the matroid $\Delta M/x'$ can be obtained from $M\backslash x$ by relabelling y and z by z' and y', respectively; and*
(iii) *each of $\Delta M/a'$, $\Delta M/b'$, and $\Delta M/c'$ is 3-connected.*

Proof. We know that each of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 3-connected. Hence $M\backslash a,b$ is connected and, by considering circuits, it is straightforward to check that ΔM is connected. Suppose ΔM has a 2-separation (X_1,X_2). Without loss of generality, we may assume that $\{a',b'\} \subseteq X_1$. If $\{a',b',c'\} \subseteq X_1$, then we can add Δ to X_1 without raising the rank, so we get a 2-separation of $P_\Delta(M(K_4),M)$; a contradiction. Thus we may assume that $c' \in X_2$. But $c' \in \mathrm{cl}^*{}_{\Delta M}(X_1)$. Hence, provided $|X_2| > 2$, we get that $(X_1 \cup c', X_2 - c')$ is a 2-separation of ΔM and $\{a',b',c'\} \subseteq X_1 \cup c'$, so we have a contradiction, as above. Hence we may assume that $|X_2| > 2$. In that case, X_2 is a series pair $\{c',x\}$ in $P_\Delta(M(K_4),M)\backslash\{a,b,c\}$. The triangle $\{a',b',c\}$ implies that $\{c',x\}$ is also a cocircuit of $P_\Delta(M(K_4),M)\backslash\{a,b\}$. It follows, by orthogonality, that $\{c',x,a,b\}$ is a cocircuit of $P_\Delta(M(K_4),M)$. Thus $\{x,a,b\}$ is a triad of M; a contradiction since $\{a,b,c\}$ is not in a fan with four or more elements. We deduce that ΔM is 3-connected.

Evidently, $\Delta M/a'$ can be obtained from $M\backslash a$ by relabelling b and c by c' and b', respectively. Thus $\Delta M/a'$ is 3-connected and, by symmetry, (ii) and (iii) hold. □

Under the relabelling described in (ii) of the last lemma, the 3-separations $(A_1,A_2),(B_1,B_2)$, and (C_1,C_2) of $M\backslash a$, $M\backslash b$, and $M\backslash c$ map to the 3-separations $((A_1-b)\cup c',(A_2-c)\cup b'),((B_1-c)\cup a',(B_2-a)\cup c')$, and $((C_1-a)\cup b',(C_2-b)\cup a')$ of $\Delta M/a'$, $\Delta M/b'$, and $\Delta M/c'$, respectively. These 3-separations are also 3-separations of the dual matroids. We shall denote these 3-separations by $(A'_1,A'_2),(B'_1,B'_2)$, and (C'_1,C'_2). The following table summarizes the inclusions we have.

a'	b'	c'
B'_1	C'_1	A'_1
C'_2	A'_2	B'_2

TABLE 2. Location of the elements of $\{a',b',c'\}$.

We shall write M' for $(\Delta M)^*$. Observe that to obtain the matroid M' in (iii) of Theorem 3.1, we need to relabel a', b', and c' as a, b, and c. However, for clarity in the remaining proofs in this section, we shall not do this relabelling yet.

Lemma 8.3. *Each of M', $M'\backslash a'$, $M'\backslash b'$, and $M'\backslash c'$ is 3-connected. Moreover, $(A'_1, A'_2), (B'_1, B'_2)$, and (C'_1, C'_2) are 3-separations in M' that are exposed by a', b', and c', respectively.*

Proof. The first sentence is an immediate consequence of the last lemma. Now suppose that $(A'_1, A'_2) \cong (A''_1, A''_2)$ in $M'\backslash a'$ and that $(A''_i \cup a', A''_j)$ is an exactly 3-separating partition of M', for some $\{i, j\} = \{1, 2\}$. Then (A'_1, A'_2) and (A''_1, A''_2) are equivalent exactly 3-separating partitions of $(M')^*/a'$, that is, of $\Delta M/a'$; and $(A''_i \cup a', A''_j)$ is an exactly 3-separating partition of ΔM. Recall that $\Delta M/a'$ is $M\backslash a$ with b and c relabelled as c' and b', respectively. Hence $M\backslash a$ has an exactly 3-separating partition (X_i, X_j) that is equivalent to (A_i, A_j) and corresponds to (A''_i, A''_j) under this relabelling. Since $a \notin \mathrm{cl}(X_i) \cup \mathrm{cl}(X_j)$, it follows that neither X_i nor X_j contains $\{b, c\}$. It follows that one of b' and c' is in A''_i and the other is in A''_j.

By (5.0.1), $|A''_j| > 3$. Now $\Delta M\backslash(A''_i \cup a')$ contains exactly one of b' and c' so this element is a coloop of $\Delta M\backslash(A''_i \cup a')$. Thus $(A''_i \cup a' \cup \{b', c'\}, A''_j - \{b', c'\})$ is an exactly 3-separating partition of ΔM and so $(A''_i \cup \{b', c'\}, A''_j - \{b', c'\})$ is an exactly 3-separating partition of $\Delta M/a'$. But the last exactly 3-separating partition is equivalent to (A''_i, A''_j) as $|A''_i \cup \{b', c'\}| = |A''_i| + 1$, so we have reduced to the case in which $\{b', c'\} \subseteq A''_i$, which we have already eliminated. We conclude that the 3-separation (A'_1, A'_2) of $M'\backslash a'$ is, indeed, exposed by a' and the rest of the lemma follows by symmetry. □

Lemma 8.4. *Assume that $|E(M)| \neq 11$, that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, and that $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. Then*
 (i) *$\lambda_{M'\backslash a',c'}(A'_1 \cap C'_2) = 2$ and $|A'_2 \cap C'_1| \geq 4$;*
 (ii) *$\lambda_{M'\backslash b',a'}(B'_1 \cap A'_2) = 2$ and $|B'_2 \cap A'_1| \geq 4$; and*
 (iii) *$\lambda_{M'\backslash c',b'}(C'_1 \cap B'_2) = 2$ and $|C'_2 \cap B'_1| \geq 4$.*

Proof. First observe that $A'_2 \cap C'_1 = (A_2 \cap C_1) \cup b'$ and $A'_1 \cap C'_2 = (A_1 \cap C_2) - b$. Since, by Lemma 8.1, $|C_2 \cap A_1| \geq 3$ and $|C_1 \cap A_2| \geq 3$, we deduce that $|A'_2 \cap C'_1| \geq 4$ and $|A'_1 \cap C'_2| \geq 2$. By symmetry, $|B'_2 \cap A'_1| \geq 4$ and $|C'_2 \cap B'_1| \geq 4$.

Now assume that none of (i), (ii), or (iii) holds. Since $M\backslash a$ is 3-connected and $A_1 \cap B_2$ is a non-minimal 3-separation of $M\backslash a, b$, it follows by Bixby's Lemma (see [1] or [6, Proposition 8.4.6]) that $M\backslash a/b$ is 3-connected up to parallel pairs. Hence $(M\backslash a/b)^*$ is 3-connected up to series pairs. But $\Delta M/a', c'$ is $M\backslash a/b$ with c relabelled as b'. Thus $M'\backslash a', c'$ is 3-connected up to series pairs.

Now $\lambda_{M'\backslash c'}(C'_2) = \lambda_{\Delta M/c'}((C_2 - b) \cup a') = \lambda_{M\backslash c}(C_2) = 2$. Likewise, $\lambda_{M'\backslash a'}(A'_1) = 2$. Thus, since $|C'_2| \geq 4$ and $M'\backslash a', c'$ is 3-connected up to series pairs, $2 \leq \lambda_{M'\backslash a',c'}(C'_2 - a') \leq 2$ so $\lambda_{M'\backslash a',c'}(C'_2 - a') = 2$. Similarly, $\lambda_{M'\backslash a',c'}(A'_1 - c') = 2$.

By the submodularity of the connectivity function,

$$4 \geq \lambda_{M'\backslash a',c'}((A'_1 - c') \cup (C'_2 - a')) + \lambda_{M'\backslash a',c'}(A'_1 \cap C'_2),$$

so $4 \geq \lambda_{M'\backslash a',c'}(A'_2 \cap C'_1) + \lambda_{M'\backslash a',c'}(A'_1 \cap C'_2)$. As $M'\backslash a', c'$ is 3-connected up to series pairs and $|A'_1 \cap C'_2| \geq 2$, we deduce that either $\lambda_{M'\backslash a',c'}(A'_1 \cap C'_2) = 2$, or $A'_1 \cap C'_2$ is a series pair of $M'\backslash a', c'$. In the first case, since $|A'_2 \cap C'_1| \geq 4$, we have that (i) holds; a contradiction. Thus $|A'_1 \cap C'_2| = 2$.

By Lemma 8.1 and symmetry, since neither (ii) nor (iii) holds, each of $|B'_1 \cap A'_2|$ and $|C'_1 \cap B'_2|$ is 2. Hence each of $|A_1 \cap C_2|$, $|B_1 \cap A_2|$, and $|C_1 \cap B_2|$ is 3. But $|A_1 \cap B_2| \geq 3$ so, by symmetry, we may assume that $|A_1 \cap B_2 \cap C_2| \geq 2$. Since $b \in C_2$, it follows that $A_1 \cap B_1 \subseteq C_1$. Now $|B_1 \cap C_2| \geq 3$, so $|B_1 \cap C_2 \cap A_2| \geq 3$. Since $A_2 \cap B_1$ also contains c, we deduce that $|A_2 \cap B_1| \geq 4$, contradicting the fact that $|A_2 \cap B_1| = 3$.

We conclude that at least one of (i), (ii), and (iii) holds. But, by applying Lemma 7.6 to M', we conclude that all of (i), (ii), and (iii) hold. □

To conclude this section, we prove three lemmas that will be used in the proof of Corollary 3.3. We also note that a trident in M yields a trident in M'.

Lemma 8.5. *Let $\{a, b, c\}$ be a wild triangle that is in a trident in a 3-connected matroid M. Then both* $\mathrm{co}(M\backslash a, b)$ *and* $\mathrm{si}(M\backslash a/b)$ *are 3-connected.*

Proof. We know that $M\backslash a$ has a quad Q containing b. By applying Lemma 2.9 to $M\backslash a$, we deduce that $\mathrm{si}(M\backslash a/b)$ is 3-connected. Now Q is also a quad of $(M\backslash a)^*$, that is, of M^*/a. By applying Lemma 2.9 to the last matroid, we deduce that $\mathrm{si}(M^*/a/b)$ is 3-connected. Thus $(\mathrm{co}(M\backslash a, b))^*$ and hence $\mathrm{co}(M\backslash a, b)$ is 3-connected. □

Lemma 8.6. *Let $\{a, b, c\}$ be a standard wild triangle in a 3-connected matroid M. Then*

(i) $\mathrm{si}(M\backslash a/b)$ *is not 3-connected; and*
(ii) $\mathrm{co}(M\backslash a, b)$ *is 3-connected.*

Proof. Let $(P_1, P_2, \ldots, P_6)$ be a partition associated to $\{a, b, c\}$. Then $(P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup c)$ is a flower. Thus $(P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup P_1 \cup P_2 \cup \{a, c\})$ is a 3-separation of M. Moreover, $b \in \mathrm{cl}(P_3 \cup P_4) \cap \mathrm{cl}(P_5 \cup P_6 \cup P_1 \cup P_2 \cup \{a, c\})$ and $|P_3|, |P_4| \geq 2$. Thus $(P_3 \cup P_4, P_5 \cup P_6 \cup P_1 \cup P_2 \cup c)$ is a vertical 2-separation of $M/b\backslash a$ unless $r_M(P_3 \cup P_4 \cup b) = 2$. But, in the exceptional case, $b \in \mathrm{cl}(P_3)$, so $(P_4 \cup P_5 \cup P_6 \cup c, P_1 \cup P_2 \cup P_3 \cup a)$ is not exposed by b; a contradiction. We deduce that $\mathrm{si}(M\backslash a/b)$ is not 3-connected. Since $M\backslash a$ is 3-connected, it follows by Bixby's Lemma that $\mathrm{co}(M\backslash a, b)$ is 3-connected. □

Lemma 8.7. *Let $\{a, b, c\}$ be a costandard wild triangle in a 3-connected matroid M. Then*

(i) $\mathrm{co}(M\backslash a,b)$ *is not* 3*-connected; and*
(ii) $\mathrm{si}(M\backslash a/b)$ *is* 3*-connected.*

Proof. Retaining the labelling we have been using in this section for ΔM, we have that $\{a',b',c'\}$ is a standard wild triangle of $(\Delta M)^*$. By Lemma 8.6 and symmetry, $\mathrm{si}((\Delta M)^*/c'\backslash b')$ is not 3-connected. It follows that $\mathrm{co}(\Delta M\backslash c'/b')$ is not 3-connected. But $\Delta M\backslash c'/b'$ is $M\backslash a,b$ with c relabelled as a'. Hence $\mathrm{co}(M\backslash a,b)$ is not 3-connected. Moreover, by Bixby's Lemma, $\mathrm{si}(M\backslash a/b)$ is 3-connected. □

Lemma 8.8. *Let $\{a,b,c\}$ be a wild triangle in a 3-connected matroid M. If $\{a,b,c\}$ is in a trident R in M, then $(R-\{a,b,c\})\cup\{a',b',c'\}$ is a trident in M'.*

Proof. Let $R=\{a,b,c,s,t,u,v\}$ where $\{t,s,u,b\},\{t,u,v,c\}$, and $\{t,s,v,a\}$ are exposed quads in $M\backslash a, M\backslash b$, and $M\backslash c$, respectively. Then one easily checks that $\{t,s,u,a',c'\},\{t,u,v,a',b'\}$, and $\{t,s,v,b',c'\}$ are circuits of ΔM; and $\{t,s,u,c'\},\{t,u,v,a'\}$, and $\{t,s,v,b'\}$ are cocircuits of ΔM. The result follows since M' is $(\Delta M)^*$. □

9. The Target

In this section, we treat case (E). We begin by noting the following immediate consequence of Lemmas 7.5 and 7.6.

Corollary 9.1. *Suppose that $\lambda_{M\backslash a,b}(A_1\cap B_2)=2$ and $|A_2\cap B_1|>3$. Then either*

(i) $|E(M)|=11$ *or* $(a,b,c,\mathbf{A},\mathbf{B},\mathbf{C})$ *is a pretrident; or*
(ii) $\lambda_{M\backslash b,c}(B_1\cap C_2)=2$ *and* $|B_2\cap C_1|>3$, *and* $\lambda_{M\backslash c,a}(C_1\cap A_2)=2$ *and* $|C_2\cap A_1|>3$.

In view of this result, many of the lemmas in this section will assume not only that case (E) occurs but also that the symmetric conditions listed in (ii) above hold.

Lemma 9.2. *If $\lambda_{M\backslash a,b}(A_1\cap B_2)=2$ and $|A_2\cap B_1|>2$, then*

(i) $c\in\mathrm{cl}((A_2\cap B_1)-c)$ *and* $c\notin\mathrm{cl}^*((A_2\cap B_1)-c)$*; and*
(ii) $|A_2\cap B_1\cap C_1|>0$ *and* $|A_2\cap B_1\cap C_2|>0$.

Proof. By (5.0.11), since $\lambda_{M\backslash a,b}(A_1\cap B_2)=2$, we have $\lambda_M(A_2\cap B_1)=2$. As $|A_2\cap B_1|>2$ and $c\in A_2\cap B_1$, Lemma 2.5 implies that $c\in\mathrm{cl}^{(*)}((A_2\cap B_1)-c)$. But $\{a,b,c\}$ is a triangle of M so, by orthogonality, $c\notin\mathrm{cl}^*((A_2\cap B_1)-c)$. Hence $c\in\mathrm{cl}((A_2\cap B_1)-c)$. Thus (i) holds. As $c\notin\mathrm{cl}(C_1)\cup\mathrm{cl}(C_2)$, it follows that $(A_2\cap B_1)-c\not\subseteq C_1$, and $(A_2\cap B_1)-c\not\subseteq C_2$, so (ii) holds. □

Next we prove a lemma about flowers that we shall need. It relies heavily on results from [7].

Lemma 9.3. *Let P be a 2-element petal of a tight flower Φ of order at least three in a 3-connected matroid N. Let (R, G) be a 3-separation of N with $|R|, |G| \geq 4$. If both $R \cap P$ and $G \cap P$ are non-empty, then Φ has order three, and the union of P with one of the other petals is a quad.*

Proof. Let $\Phi = (P_1, P_2, \dots, P_n)$ and $P_2 = P$. Because Φ has order at least three, all the petals of Φ are tight. Each element of P is tight since P must contain at least one tight element but, by [7, Lemma 5.8], P cannot contain exactly one tight element. Let $P_2 \cap R = \{r_2\}$ and $P_2 \cap G = \{g_2\}$.

Suppose that $|P_1 \cap G| \geq 2$ and $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap R| \geq 2$. Then $r_2 \cup (P_3 \cup P_4 \cup \dots \cup P_n)$, which is the union of the 3-separating sets $P_3 \cup P_4 \cup \dots \cup P_n$ and $R \cap (P_2 \cup P_3 \cup \dots \cup P_n)$, is also 3-separating. Hence $r_2 \in \mathrm{fcl}(P_3 \cup P_4 \cup \dots \cup P_n)$ so, by [7, Lemma 5.9], r_2 is loose, a contradiction. We deduce that $|P_1 \cap G| \leq 1$ or $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap R| \leq 1$. By symmetry, $|P_1 \cap R| \leq 1$ or $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap G| \leq 1$. Because $|G| \geq 4$ and $|R \cap P_2|, |G \cap P_2| = 1$, at most one of $|P_1 \cap G| \leq 1$ and $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap G| \leq 1$ holds. By the symmetry between R and G, we deduce that either $|P_1 \cap G| \leq 1$ and $|P_1 \cap R| \leq 1$; or $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap G| \leq 1$ and $|(P_3 \cup P_4 \cup \dots \cup P_n) \cap R| \leq 1$. By reflective symmetry in Φ, we also have that either $|P_3 \cap G| \leq 1$ and $|P_3 \cap R| \leq 1$; or $|(P_4 \cup P_5 \cup \dots \cup P_n \cup P_1) \cap G| \leq 1$ and $|(P_4 \cup P_5 \cup \dots \cup P_n \cup P_1) \cap R| \leq 1$. Since $|R|, |G| \geq 4$, it follows that we must have $n = 3$ and either P_1 or P_3 has exactly two elements.

Suppose that $|P_1| = 2$. Then $P_1 \cup P_2$ is 3-separating in N having exactly four elements. If $P_1 \cup P_2$ properly contains a circuit or a cocircuit, then P_1 or P_2 is not tight. Hence $P_1 \cup P_2$ is a quad. □

Lemma 9.4. *Assume that $\lambda_{M\backslash a,b}(A_1 \cap B_2) = \lambda_{M\backslash b,c}(B_1 \cap C_2) = \lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$ and $|A_2 \cap B_1|, |B_2 \cap C_1|, |C_2 \cap A_1| > 3$. Then (C_1, C_2) can be replaced by an equivalent 3-separating partition exposed by c such that*

$$A_1 \cap B_1 \cap C_1 = \emptyset = A_2 \cap B_2 \cap C_2.$$

Proof. First observe that, by (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. Now, by Lemma 9.2, $A_2 \cap B_1 \cap C_1$ and $A_2 \cap B_1 \cap C_2$ are non-empty. Suppose that $A_2 \cap B_1 \cap C_1 = \{r_{21}\}$. Then $|(A_2 \cap B_1) - c| \geq 3$ and $\lambda_{M\backslash c}((A_2 \cap B_1) - c) = 2$, so $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_1 \cap C_2)$. Hence $(C_1 - r_{21}, C_2 \cup r_{21}) \cong (C_1, C_2)$. But $c \in \mathrm{cl}((A_2 \cap B_1 \cap C_2) \cup r_{21}) \subseteq \mathrm{cl}(C_2 \cup r_{21})$ contradicting the fact that c exposes (C_1, C_2). We deduce that

$$|A_2 \cap B_1 \cap C_1| \geq 2.$$

A symmetric argument to that just given establishes that

$$|A_2 \cap B_1 \cap C_2| \geq 2.$$

The next observation simplifies the argument to follow.

9.4.1. *If $|A_1 \cap B_1 \cap C_2| = 1$, then $|A_1 \cap B_1 \cap C_1| = 1$.*

Assume that $|A_1 \cap B_1 \cap C_1| \geq 2$ and let $A_1 \cap B_1 \cap C_2 = \{g_{11}\}$. As $\lambda_{M\backslash c}(A_1 \cap B_1) = 2$, we deduce that $g_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}((A_1 \cap B_1) - g_{11})$ so we can replace (C_1, C_2) by the equivalent $(C_1 \cup g_{11}, C_2 - g_{11})$. After this is done, $A_1 \cap B_1 \subseteq C_1$. But, by Lemma 9.2 and symmetry, $b \in \mathrm{cl}((A_1 \cap C_2) - b)$. Hence $b \in \mathrm{cl}(B_2)$; a contradiction. Thus (9.4.1) holds.

Most of the rest of the proof of the lemma will be occupied with proving the following assertion from which the lemma will follow straightforwardly.

9.4.2. *(C_1, C_2) can be replaced by an equivalent 3-separation in which $A_1 \cap B_1 \cap C_1 = \emptyset$ and $A_2 \cap B_2 \cap C_2$ is unchanged.*

Suppose not and assume that $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$. If $|A_1 \cap B_1 \cap C_2| \geq 2$, then $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}((A_1 \cap B_1) - r_{11})$ and we can replace (C_1, C_2) by $(C_1 - r_{11}, C_2 \cup r_{11})$ to obtain that $A_1 \cap B_1 \cap C_1 = \emptyset$. Since this change has no effect on $A_2 \cap B_2 \cap C_2$, we have a contradiction. Hence if $|A_1 \cap B_1 \cap C_1| = 1$, then we may assume that $|A_1 \cap B_1 \cap C_2| = 1$. On the other hand, if $|A_1 \cap B_1 \cap C_1| \geq 2$, then, by (9.4.1) and the consequence of Lemma 9.2 that $A_1 \cap B_1 \cap C_2$ is non-empty, we deduce that $|A_1 \cap B_1 \cap C_2| \geq 2$.

We show next that

9.4.3. *$|A_1 \cap B_1 \cap C_1| \geq 2$ and $|A_2 \cap B_2 \cap C_2| \geq 2$.*

From above, we know that if (9.4.3) fails, then we may assume that

$$|A_1 \cap B_1 \cap C_1| = 1 = |A_1 \cap B_1 \cap C_2|,$$

so $|A_1 \cap B_1| = 2$.

Consider the flower $(B_2, A_1 \cap B_1, A_2 \cap B_1)$ in $M\backslash b$. We show next that this flower, Φ_b, is tight. Assume not. The petal B_2 is not loose otherwise $B_2 \subseteq \mathrm{fcl}_{M\backslash b}(B_1)$ and (B_1, B_2) is sequential; a contradiction. If $A_2 \cap B_1$ is loose, then, by Lemma 2.10, $\mathrm{fcl}_{M\backslash b}(A_1 \cap B_1) \supseteq A_2 \cap B_1$. But $|A_1 \cap B_1| = 2$ and so B_1 is sequential; a contradiction. Hence $A_2 \cap B_1$ is not loose.

Now suppose that $A_1 \cap B_1$ is loose. By Lemma 2.10, $A_1 \cap B_1 \subseteq \mathrm{fcl}(B_2)$ so (B_1, B_2) can be replaced by an equivalent 3-separating partition in which $|A_1 \cap B_1| < 2$; a contradiction. We deduce that $A_1 \cap B_1$ is not loose, so Φ_b is a tight flower. Therefore, by Lemma 9.3, $|A_2 \cap B_1| = 2$ or $|B_2| = 2$. Neither of these holds, and this contradiction completes the proof that (9.4.3) holds.

To complete the proof of (9.4.2), we shall apply Lemma 8.2 of [7] to the flower Φ_b using the 3-separation $(B_1 \cap C_1, E - b - (B_1 \cap C_1))$ as (R, G) of that lemma. By that lemma, there is a flower Φ'_b that refines Φ_b and displays $(B_1 \cap C_1, E - b - (B_1 \cap C_1))$, namely $(B_2, A_1 \cap B_1 \cap C_2, A_1 \cap B_1 \cap C_1, A_2 \cap B_1 \cap C_1, (A_2 \cap B_1) - C_1)$. Let $Z = (E - b) - (A_2 \cap B_1)$. Then $a \in B_2 \subseteq Z$. Since $A_1 - b \subseteq Z$, we deduce that $b \in \mathrm{cl}(Z)$. Hence $\{a, b\} \subseteq \mathrm{cl}(Z)$, so $c \in \mathrm{cl}(Z)$. By applying [7, Lemma 5.5(ii)] to Φ'_b, we deduce that $c \in \mathrm{cl}(B_2)$ or $c \in \mathrm{cl}(A_1 \cap B_1 \cap C_1)$. But $a \in B_2$. If $c \in \mathrm{cl}(B_2)$, then $b \in \mathrm{cl}(B_2)$; a contradiction. Hence $c \in \mathrm{cl}(A_1 \cap B_1 \cap C_1)$, so $c \in \mathrm{cl}(C_1)$; a contradiction.

We conclude that (9.4.2) holds. Using a symmetric argument, we can modify (C_1, C_2) again to get $A_2 \cap B_2 \cap C_2 = \emptyset$ while maintaining $A_1 \cap B_1 \cap C_1 = \emptyset$. $\square$

The next lemma completes the treatment of case (E) by showing that when (E) and the two sets of symmetric conditions hold, $\{a, b, c\}$ is a standard wild triangle in M and $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ is an associated partition.

Lemma 9.5. *Assume that* $\lambda_{M\backslash a,b}(A_1 \cap B_2) = \lambda_{M\backslash b,c}(B_1 \cap C_2) = \lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$ *and* $|A_2 \cap B_1|, |B_2 \cap C_1|, |C_2 \cap A_1| > 3$. *If* $A_1 \cap B_1 \cap C_1 = \emptyset = A_2 \cap B_2 \cap C_2$, *then*

(i) $(A_2 \cap B_1, B_2 \cap C_1, C_2 \cap A_1)$ *is a flower in* M;
(ii) $M\backslash a, b, c$ *is connected,* $\mathrm{co}(M\backslash a, b, c)$ *is 3-connected, and every 2-element cocircuit of* $M\backslash a, b, c$ *meets exactly two of* $A_2 \cap B_1, B_2 \cap C_1$, *and* $C_2 \cap A_1$; *and*
(iii) $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ *partitions the ground set of* $M\backslash a, b, c$ *and every union of consecutive sets is exactly 3-separating in* $M\backslash a, b, c$.

Proof. Since $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$, by (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. By symmetry, $\lambda_M(B_2 \cap C_1) = \lambda_M(C_2 \cap A_1) = 2$. Hence $(A_2 \cap B_1, B_2 \cap C_1, C_2 \cap A_1)$ is a flower Ψ in M.

Certainly $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ partitions $E(M\backslash a, b, c)$. Moreover, by (5.0.6) and (5.0.7), all six of the sets in the partition are exactly 3-separating in $M\backslash a, b, c$. The unions of the first and second sets and of the second and third sets are $B_2 \cap C_1$ and $A_1 \cap B_2$, respectively. Both of the last two sets are exactly 3-separating in $M\backslash a, b, c$ since $\lambda_M(B_2 \cap C_1) = 2 = \lambda_{M\backslash a,b,c}(B_2 \cap C_1)$ and $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2 = \lambda_{M\backslash a,b,c}(A_1 \cap B_2)$. By symmetry, we deduce that the union of every two consecutive sets in the distinguished partition is exactly 3-separating. The union of the first three sets is $B_2 - a$, which is exactly 3-separating in $M\backslash a, b, c$. By using symmetry and taking complements, we deduce that every union of consecutive sets in the distinguished partition is exactly 3-separating.

We show next that

9.5.1. $M\backslash a, b, c$ *is connected.*

Since each of $M\backslash a, M\backslash b$, and $M\backslash c$ is 3-connected, each of $M\backslash a, b$, $M\backslash b, c$ and $M\backslash c, a$ is connected. Suppose that (X, Y) is a 1-separation in $M\backslash a, b, c$ and consider the flower Ψ of M. If $X \supseteq (A_1 \cap C_2) - b$, then $(X \cup b, Y)$ is a 1-separation of $M\backslash a, c$; a contradiction. Thus, for every 1-separation (X, Y) of $M\backslash a, b, c$, each of X and Y meets every petal of Ψ. Now each such petal has at least four elements. Without loss of generality, $X \cap (A_1 \cap C_2)$ has at least two elements. Since this set is contained in $A_1 \cap C_2$, we have

$\lambda_{M\backslash a,b,c}(X \cap A_1 \cap C_2) = \lambda_M(X \cap A_1 \cap C_2) \geq 2$. Now

$$\begin{aligned} 2 &= \lambda_{M\backslash a,b,c}(X) + \lambda_{M\backslash a,b,c}((A_1 \cap C_2) - b) \\ &\geq \lambda_{M\backslash a,b,c}(X \cap A_1 \cap C_2) + \lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)) \\ &\geq 2 + \lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)). \end{aligned}$$

Hence $\lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)) = 0$, so $(X \cup ((A_1 \cap C_2) - b), Y - ((A_1 \cap C_2) - b))$ is a 1-separation of $M\backslash a, b, c$ in which $Y - ((A_1 \cap C_2) - b)$ avoids some petal of Ψ; a contradiction. We deduce that (9.5.1) holds.

Now let (X, Y) be a 2-separation of $M\backslash a, b, c$. We show next that:

9.5.2. *If $X \supseteq (A_1 \cap C_2) - b$, then $|Y \cap A_2 \cap B_1| = 1 = |Y \cap B_2 \cap C_1|$ so $|Y| = 2$.*

Suppose that $X \supseteq (A_1 \cap C_2) - b$. Then $(X \cup b, Y)$ is a 2-separation of $M\backslash a, c$. As $M\backslash a$ and $M\backslash c$ are 3-connected, neither X nor Y contains $(B_2 \cap C_1) - a$ or $(A_2 \cap B_1) - c$. Now $\lambda_{M\backslash a,c}(X \cup b) = 1$ and $\lambda_{M\backslash a,c}((B_2 \cap C_1) - a) = 2$. Thus, by the submodularity of the connectivity function,

$$\begin{aligned} 3 &\geq \lambda_{M\backslash a,c}(X \cup b \cup ((B_2 \cap C_1) - a)) + \lambda_{M\backslash a,c}((X \cup b) \cap ((B_2 \cap C_1) - a)) \\ &= \lambda_{M\backslash c}(X \cup b \cup (B_2 \cap C_1)) + \lambda_{M\backslash a}(X \cap B_2 \cap C_1) \\ &= \lambda_{M\backslash c}(Y \cap A_2 \cap B_1) + \lambda_{M\backslash a}(X \cap B_2 \cap C_1). \end{aligned}$$

Since both $M\backslash c$ and $M\backslash a$ are 3-connected and both $Y \cap A_2 \cap B_1$ and $X \cap B_2 \cap C_1$ are nonempty, we deduce that

$$|Y \cap A_2 \cap B_1| = 1 \text{ or } |X \cap B_2 \cap C_1| = 1.$$

By symmetry,

$$|Y \cap B_2 \cap C_1| = 1 \text{ or } |X \cap A_2 \cap B_1| = 1.$$

Since both $|(A_2 \cap B_1) \cap (X \cup Y)|$ and $|(B_2 \cap C_1) \cap (X \cup Y)|$ exceed two, we deduce that either $|Y \cap A_2 \cap B_1| = 1 = |Y \cap B_2 \cap C_1|$, or $|X \cap A_2 \cap B_1| = 1 = |X \cap B_2 \cap C_1|$. In the first case, the required result holds so assume that the second case occurs letting x be the unique element of $B_2 \cap C_1 \cap X$. We have, by submodularity, that

$$\begin{aligned} 3 &= \lambda_{M\backslash a,b,c}(X) + \lambda_{M\backslash a,b,c}(E - \{a, b, c\} - (B_2 \cap C_1)) \\ &\geq \lambda_{M\backslash a,b,c}(X - x) + \lambda_{M\backslash a,b,c}((E - \{a, b, c\} - (B_2 \cap C_1)) \cup x) \\ &= \lambda_{M\backslash a,b,c}(X - x) + \lambda_{M\backslash a}((B_2 \cap C_1) - x - a) \\ &\geq \lambda_{M\backslash a,b,c}(X - x) + 2, \end{aligned}$$

where the last inequality holds because $|(B_2 \cap C_1) - x - a| \geq 2$. Hence $\lambda_{M\backslash a,b,c}(X - x) \leq 1$. But $X - x$ spans b and $Y \cup x$ spans a. Therefore $((X - x) \cup b, Y \cup x \cup a)$ is a 1-separation of $M\backslash c$. This contradiction completes the proof of (9.5.2).

Next we establish the following:

9.5.3. *If $|X| \geq |Y|$, then either*

(i) $|Y| = 2$ *and* Y *meets exactly two of* $A_1 \cap C_2, B_1 \cap A_2,$ *and* $C_1 \cap B_2$; *or*

(ii) $|Y| = 3$ *and* Y *meets each of* $A_1 \cap C_2, B_1 \cap A_2,$ *and* $C_1 \cap B_2$.

If $|Y| = 2$ and $Y \subseteq A_1 \cap C_2$, then $(X \cup \{a, c\}, Y)$ is a 2-separation of $M\backslash b$; a contradiction. It follows by symmetry that if $|Y| = 2$, then (i) holds. Now suppose that $|Y| \geq 3$ but (ii) does not hold. Then, by (9.5.2), both X and Y meet each of $A_1 \cap C_2, B_1 \cap A_2$, and $C_1 \cap B_2$.

Since (ii) fails, we may assume, by symmetry, that $|X \cap B_2 \cap C_1| \geq 2$. Now $\lambda_{M\backslash a,b,c}(X) = 1$ and $\lambda_{M\backslash a,b,c}((B_2 \cap C_1) - a) = 2$. Thus, by Lemma 2.2, $\lambda_{M\backslash a,b,c}(X \cap B_2 \cap C_1) = 1$, or $\lambda_{M\backslash a,b,c}(X \cup ((B_2 \cap C_1) - a)) = 1$. In each case, we have a new 2-separation of $M\backslash a, b, c$ to which we can apply one of the symmetric versions of (9.5.2). In the first case, because $|E - \{a, b, c\} - (X \cap B_2 \cap C_1)| \geq 3$, we get an immediate contradiction to (9.5.2). In the second case, (9.5.2) implies that Y contains exactly one element of each of $A_2 \cap B_1$ and $C_2 \cap A_1$. Hence $|X \cap A_2 \cap B_1| \geq 2$ so, by a symmetric argument to that just given, we get that Y contains exactly one element of each of $B_2 \cap C_1$ and $C_2 \cap A_1$. Thus (ii) holds; a contradiction. Hence (9.5.3) holds.

By (9.5.3), to complete the proof of the lemma, we need to show that if $|Y| = 3$, then Y is a series class of $M\backslash a, b, c$. Let $Y = \{y_1, y_2, y_3\}$ and suppose that Y is not a series class of $M\backslash a, b, c$. Since $r_{M\backslash a,b,c}(Y) + r^*_{M\backslash a,b,c}(Y) - |Y| = 1$, it follows that Y is a triangle of M. Moreover, Y contains a unique cocircuit of $M\backslash a, b, c$. Without loss of generality, we may assume that this cocircuit is either Y or $\{y_1, y_2\}$. We may also assume that $y_1 \in C_1 \cap B_2$, $y_2 \in A_1 \cap C_2$, and $y_3 \in B_1 \cap A_2$. We now think in terms of the familiar Venn diagram involving A_1, A_2, B_1, and B_2. Without loss of generality, $y_3 \in C_2$.

We show next that

9.5.4. $y_1 \in A_1 \cap B_2$.

We have $\lambda_{M\backslash a,b,c}(Y) = 1$ and $\lambda_{M\backslash a,b,c}(A_2 - c) = 1$. Thus, by Lemma 2.2, either $\lambda_{M\backslash a,b,c}(Y \cap (A_2 - c)) = 1$, or $\lambda_{M\backslash a,b,c}(Y \cup (A_2 - c)) = 1$. In the first case, since $Y \cap (A_2 - c)$ is y_3 or $\{y_1, y_3\}$, we deduce that $y_1 \in A_1 \cap B_2$ or $\{y_1, y_3\}$ is a cocircuit of $M\backslash a, b, c$. But the last possibility does not arise so, in the first case, $y_1 \in A_1 \cap B_2$. In the second case, $\lambda_{M\backslash a,b,c}(A_1 - b - Y) = 1$. Since $Y \cup (A_2 - c) \supseteq (A_2 \cap B_1) - c$, by (9.5.2) and symmetry, $|A_1 - b - Y| = 2$. But $|A_1 \cap B_2|, |A_1 \cap B_1| \geq 2$. Hence, again, $y_1 \in A_1 \cap B_2$. Thus (9.5.4) holds.

We now have that both y_1 and y_2 are in A_1. Thus we can move y_3 from $A_2 \cap B_1$ into $A_1 \cap B_1$ maintaining the fact that $A_1 \cap B_1 \cap C_1 = \emptyset$ but changing $|A_2 \cap B_1 \cap Y|$ to 0. This gives a contradiction to (9.5.3) provided we are still in case (E), that is, provided this move maintains the fact that $|A_2 \cap B_1| = 4$. Thus suppose that, before the move, $|(A_2 \cap B_1) - y_3| = 3$. Then, by Lemma 7.5, $(A_2 \cap B_1) - y_3$ is a triangle. Since $\lambda_M(A_1 \cup B_2) = \lambda_M(A_1 \cup B_2 \cup y_3)$ and $y_3 \in \mathrm{cl}(A_1)$, we deduce that $y_3 \in \mathrm{cl}((A_2 \cap B_1) - y_3)$. Therefore $A_2 \cap B_1$ is a 4-point line containing c. Since C_1 or C_2 must contain at least two elements of this line, we conclude that C_1 or C_2 spans c. This contradiction completes the proof of the lemma. □

We are now ready to complete the proofs of Theorems 3.1 and 1.1 and of Corollary 3.3. It should be noted here that the proof that we give for the first of these results also proves the variant of that result in which outcome (i) is replaced by the outcome that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident in M. The reason for our interest in this alternative statement is that it indicates what can be said about the structure around the wild triangle $\{a, b, c\}$ when we begin with a certain collection of 3-separations exposed by a, b, and c and only allow ourselves to move to equivalent 3-separations.

Proof of Theorem 3.1. By Corollary 4.3, we know that $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. In Corollary 7.8, we showed that if one of cases (A)-(D) arises, then $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident in M. Corollary 9.1 gives that when case (E) occurs, either $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, or we may assume that we have symmetry between $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2). In the latter case, Lemma 9.5 establishes that outcome (ii) of the theorem holds. If case (F) occurs and $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, then, by Lemma 8.4, case (E) and the symmetric conditions hold in M'. Hence outcome (iii) of the theorem arises.

Next, we note that it is shown in Lemma 7.9 that if $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, then $\{a, b, c\}$ is in a trident in M, so outcome (i) of the theorem occurs.

We now know that $\{a, b, c\}$ is in a trident, $\{a, b, c\}$ is a standard wild triangle, or $\{a, b, c\}$ is a costandard wild triangle. By combining Lemmas 8.5, 8.6, and 8.7, we obtain that these three possibilities are mutually exclusive. We conclude that exactly one of outcomes (i)-(iii) of the theorem occurs. □

Proof of Theorem 1.1. This theorem follows immediately by combining Corollary 4.3 with Theorem 3.1. □

Proof of Corollary 3.3. By Corollary 4.3, $\{a, b, c\}$ is an internal triangle of a fan if and only if $M\backslash a$ is not 3-connected. If $M\backslash a$ is 3-connected, then the rest of the corollary follows by combining Theorem 3.1 with Lemmas 8.5, 8.6, and 8.7. □

10. Finishing Off

In this section we give the proof of Corollary 3.2. The arguments here rely heavily on results from [7]. We begin with four lemmas on flowers, omitting the straightforward proof of the first. In fact, it is the dual of the first lemma that we need, but the result is more obvious in the form given.

Lemma 10.1. *Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ in a 3-connected matroid M, where $n \geq 3$, and let x be an element of P_1. Let M' be obtained by adding a nonempty set X of elements in parallel to x. Assume that there is a partition $(X_2, X_3, \ldots, X_n)$ of X, where some subsets may be empty, such* *that each union of consecutive pairs in the partition* $(P_1, P_2 \cup X_2, \dots, P_n \cup X_n)$ *of* $E(M')$ *is 3-separating. Then the following hold.*

(i) *If* $X_i \neq \emptyset$*, then* $x \in \mathrm{cl}(P_i)$ *for all* i *in* $\{1, 2, \dots, n\}$.
(ii) Φ *is not a copaddle.*
(iii) *If* Φ *is swirl-like, then, up to relabelling,* $X = X_2$ *and* x *is a loose element in* $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)$.
(iv) *If* Φ *is spike-like, then* x *is the unique element of* M *that is in* $\mathrm{cl}(P_i)$ *for all* i *in* $\{1, 2, \dots, n\}$.
(v) *If* Φ *is a paddle, then* $x \in \mathrm{cl}(P_i)$ *for all* i *in* $\{1, 2, \dots, n\}$.

Lemma 10.2. *Let* Φ *be a tight flower* (P_1, P_2, P_3, P_4) *of a 3-connected matroid* M. *Then there is a tight flower* (Q_1, Q_2, Q_3, Q_4) *equivalent to* Φ *such that* $Q_1 \cup Q_2$, Q_2, *and* $Q_2 \cup Q_3$ *are fully closed.*

Proof. Begin by considering the flower $\Phi' = (P'_1, P'_2, P'_3, P'_4) = (P_1 - \mathrm{fcl}(P_2), \mathrm{fcl}(P_2), P_3 - \mathrm{fcl}(P_2), P_4 - \mathrm{fcl}(P_2))$. By [7, Corollary 5.12 and Theorem 6.5], Φ' is a tight flower equivalent to and therefore of the same type as Φ. By [7, Lemma 5.9], $\mathrm{fcl}(P'_1 \cup P'_2) - (P'_1 \cup P'_2) \subseteq (\mathrm{fcl}(P'_1) - P'_1) \cup (\mathrm{fcl}(P'_2) - P'_2)$. But P'_2 is fully closed, so $\mathrm{fcl}(P'_1 \cup P'_2) - (P'_1 \cup P'_2) \subseteq (\mathrm{fcl}(P'_1) - P'_1)$. Using this fact and symmetry, we deduce by [7, Theorems 6.1 and 7.1] that if Φ' is Vámos-like or is an anemone, all loose elements of Φ' are contained in P'_2, so Φ' is the required flower (Q_1, Q_2, Q_3, Q_4). If Φ' is swirl-like, then, by [7, Theorem 7.4], the elements in $\mathrm{fcl}(P'_1) - (P'_1 \cup P'_2)$ and $\mathrm{fcl}(P'_3) - (P'_3 \cup P'_2)$ form disjoint subsets of P'_4. By moving these subsets of P'_4 into P'_1 and P'_3, respectively, we obtain the required flower (Q_1, Q_2, Q_3, Q_4). □

Note that there are various generalizations of the last lemma for flowers with more petals.

Lemma 10.3. *Let* Φ *be a tight flower* (P_1, P_2, P_3, P_4) *of a 3-connected matroid* M *and let* Ψ *be a partition* (Q_1, Q_2, Q_3, Q_4) *of* $E(M)$ *where* $Q_1 \cup Q_2$ *and* $Q_2 \cup Q_3$ *are 3-separating sets equivalent to* $P_1 \cup P_2$ *and* $P_2 \cup P_3$, *respectively. Then* Ψ *is a tight flower equivalent to* Φ.

Proof. By Lemma 10.2, there is a tight flower (P'_1, P'_2, P'_3, P'_4) equivalent to Φ such that $P'_1 \cup P'_2$ and $P'_2 \cup P'_3$ are fully closed. Thus

$$\begin{aligned} Q_1 \cup Q_2 \cup Q_3 &\subseteq \mathrm{fcl}(Q_1 \cup Q_2) \cup \mathrm{fcl}(Q_2 \cup Q_3) \\ &= \mathrm{fcl}(P_1 \cup P_2) \cup \mathrm{fcl}(P_2 \cup P_3) \\ &= \mathrm{fcl}(P'_1 \cup P'_2) \cup \mathrm{fcl}(P'_2 \cup P'_3) \\ &= P'_1 \cup P'_2 \cup P'_3 \end{aligned}$$

Since $|P'_4| \geq 2$, we see that $|Q_4| \geq 2$. Now $Q_3 \cup Q_4$ and $Q_4 \cup Q_1$, as the complements of $Q_1 \cup Q_2$ and $Q_2 \cup Q_3$, are also 3-separating and are equivalent to $P_3 \cup P_4$ and $P_4 \cup P_1$. Hence, by a symmetric argument to the above, we deduce that $|Q_2| \geq 2$ and, similarly, $|Q_1|, |Q_3| \geq 2$. It now follows by uncrossing that Ψ is a flower.

By Lemma 10.2 again, there is a flower (Q'_1, Q'_2, Q'_3, Q'_4) equivalent to Ψ such that $Q'_1 \cup Q'_2$ and $Q'_2 \cup Q'_3$ are fully closed. This means that $Q'_1 \cup Q'_2 = P'_1 \cup P'_2$ and $Q'_2 \cup Q'_3 = P'_2 \cup P'_3$. Hence $(Q'_1, Q'_2, Q'_3, Q'_4) = (P'_1, P'_2, P'_3, P'_4)$. Thus Ψ is equivalent to Φ. □

The last lemma proves the base case of the following more general result.

Lemma 10.4. *Let Φ be a tight flower $(P_1, P_2, \dots, P_k)$ in a 3-connected matroid M, let Ψ be a partition of $(Q_1, Q_2, \dots, Q_k)$ of $E(M)$, and let t be an integer with $2 \leq t \leq k-2$. Assume that, for all j in $\{1, 2, \dots, k\}$, the set $Q_{j+1} \cup Q_{j+2} \cup \dots \cup Q_{j+t}$ is an exactly 3-separating set equivalent to $P_{j+1} \cup P_{j+2} \cup \dots \cup P_{j+t}$. Then Ψ is a flower equivalent to Φ.*

Proof. Let $C(t)$ be the specified condition on the sets $Q_{j+1} \cup Q_{j+2} \cup \dots \cup Q_{j+t}$. Evidently, if $C(s)$ holds, then so does $C(k-s)$. Now suppose that $C(s)$ holds for some s with $2 < s \leq k-2$. By [7, Lemma 5.9], the flower $(P_1, P_2 \cup \dots \cup P_s, P_{s+1}, P_{s+2} \cup \dots \cup P_k)$ is tight. By applying Lemma 10.3 to this flower and the partition $(Q_1, Q_2 \cup \dots \cup Q_s, Q_{s+1}, Q_{s+2} \cup \dots \cup Q_k)$, we deduce that $Q_2 \cup \dots \cup Q_s$ is an exactly 3-separating set equivalent to $P_2 \cup \dots \cup P_s$. Hence, by symmetry, if $C(s)$ holds, then so does $C(s-1)$. Since $C(t)$ holds, we deduce that $C(s)$ holds for all s in $\{2, 3, \dots, k-2\}$. Since $C(2)$ holds, Ψ is a flower.

Consider the flowers $\Phi, (P_1, P_2, P_3, P_4 \cup \dots \cup P_k), (Q_1, Q_2, Q_3, Q_4 \cup \dots \cup Q_k)$, and Ψ. By Lemma 10.3, the second and third are equivalent. Moreover, if one is an anemone, they all are. Assume that Φ is a daisy. Then so is Ψ. Moreover, as $C(s)$ holds for all s in $\{2, 3, \dots, k-2\}$, it follows that Ψ is equivalent to Φ.

Finally, assume that Φ is an anemone. Then the tight flower $(P_1, P_3, P_2, P_4, P_5, \dots, P_k)$ and the partition $(Q_1, Q_3, Q_2, Q_4, Q_5, \dots, Q_k)$ obey $C(2)$ and hence obey $C(s)$ for all s in $\{2, 3, \dots, k-2\}$. As every permutation of $(1, 2, \dots, k)$ can be obtained as a product of transpositions, it is now straightforward to deduce that the anemones Φ and Ψ are equivalent. □

Proof of Corollary 3.2. By the definition of an associated partition, the partition $\mathbf{P}$ is associated with 3-separations (A'_1, A'_2), (B'_1, B'_2), and (C'_1, C'_2) exposed by a, b, and c, respectively. By Theorem 3.1, there are equivalent exposed 3-separations (A_1, A_2), (B_1, B_2), and (C_1, C_2) that satisfy (a)-(c) of part (ii) of that theorem. Recall that $N = \mathrm{co}(M \backslash a, b, c)$ and that $\mathbf{Q}$ is the partition $(Q_1, Q_2, \dots, Q_6)$ of $E(N)$ induced by $\mathbf{P}$. Let $(R_1, R_2, \dots, R_6)$ be the partition $\mathbf{R}$ of $E(N)$ induced by $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$.

10.4.1. $\mathbf{R}$ *is a tight flower in* N.

From Theorem 3.1(ii)(c), all unions of consecutive sets of $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ are 3-separating in $M \backslash a, b, c$. Hence all unions of consecutive sets of $\mathbf{R}$ are 3-separating in N as, by Lemma 2.3, the connectivity of a set cannot increase in a minor. To show that $\mathbf{R}$ is a tight flower in N, it suffices, by Lemma 2.10 and symmetry, to show that R_1 contains at least two elements that are not in the full closure of R_6. Recall that $R_1 = (A_2 \cap B_2) \cap E(N)$.

Since $A_2 \cap B_2$, $A_1 \cap C_1$, and their union are 3-separating in $M\backslash a$, it follows that $(A_2 \cap B_2, A_1 \cap C_1, B_1 \cup C_2)$ is a flower in $M\backslash a$. We show next that this flower is tight. If not, then, by Lemma 2.10 and symmetry, either $B_1 \cup C_2 \subseteq \mathrm{fcl}_{M\backslash a}(A_2 \cap B_2)$, or $A_2 \cap B_2 \subseteq \mathrm{fcl}_{M\backslash a}(A_1 \cap C_1)$. In the first case, $A_2 \cap B_1 \subseteq \mathrm{fcl}_{M\backslash a}(A_2 \cap B_2)$, so the elements of $A_2 \cap B_1$ can be moved from B_1 into B_2 to make $A_2 \cap B_1$ empty, contradicting (5.0.9). In the second case, we can move the elements of $A_2 \cap B_2$ into A_1 to reduce $A_2 \cap B_2$ to an empty set, contradicting (5.0.5). Thus $(A_2 \cap B_2, A_1 \cap C_1, B_1 \cup C_2)$ is, indeed, a tight flower in $M\backslash a$.

Since $A_2 \cap B_2$ contains at least two tight elements of this flower, to complete the proof of (10.4.1), it suffices to show that if x is such an element, then $x \notin \mathrm{fcl}_N(R_6)$. Evidently, $x \notin \mathrm{fcl}_{M\backslash a}(B_1 \cup C_2)$. Since $c \in \mathrm{cl}(B_1 - c)$ and $b \in \mathrm{cl}(C_2 - b)$, we deduce that $x \notin \mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$. By Theorem 3.1(ii)(b), $\mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$ contains all non-trivial series classes of $M\backslash a, b, c$, so $\mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$ contains all the elements of $M\backslash a, b, c$ that are removed to obtain N. Hence $x \in E(N)$. Moreover, as $R_3 \cup R_4 \cup R_5 \cup R_6 = [(B_1 \cup C_2) - \{b, c\}] \cap E(N)$, it follows that $x \notin \mathrm{fcl}_N(R_3 \cup R_4 \cup \cdots \cup R_6)$. In particular, $x \notin \mathrm{fcl}_N(R_6)$ and 10.4.1 follows.

Next we consider the type of the flower $\mathbf{R}$. First suppose that $N = M\backslash a, b, c$. If $\mathbf{R}$ is a paddle, then so is $(A_2 \cap B_2, (C_1 \cap A_1) \cup (B_2 \cap C_2), A_1 \cap B_1, (C_2 \cap A_2) \cup (B_1 \cap C_1))$. Now $(C_2 \cap A_2) \cup (B_1 \cap C_1) = (A_2 \cap B_1) - c$, and c is in the closure of both this set and its complement in $E(M\backslash a, b, c)$. Since $\mathbf{R}$ is a paddle, it follows that $c \in \mathrm{cl}(A_1 \cap B_1)$. Hence $\{b, c\} \subseteq \mathrm{cl}(A_1)$ so $a \in \mathrm{cl}(A_1)$; a contradiction. We deduce that $\mathbf{R}$ is not a paddle, so $\mathbf{R}$ is swirl-like, spike-like, or a copaddle, and (i)–(iii) hold for $\mathbf{R}$.

Now suppose that $N \neq M\backslash a, b, c$. Then, by combining Theorem 3.1(ii)(b) with the dual of Lemma 10.1, it is straightforward to show that $\mathbf{R}$ is not a paddle and that it satisfies (i)–(iii) of the corollary.

10.4.2. *The partition $(Q_2 \cup Q_3 \cup Q_4, Q_5 \cup Q_6 \cup Q_1)$ of $E(N)$ induced by (A'_1, A'_2) is a 3-separation equivalent to $(R_2 \cup R_3 \cup R_4, R_5 \cup R_6 \cup R_1)$.*

To see this, consider the definition of equivalence of 3-separating sets. By that, there is a sequence $(S_1, S_2, \dots, S_k)$ of sets in $M\backslash a$ with $A_1 = S_1$ and $S_k = A'_1$ such that, for all i in $\{2, 3, \dots, k\}$, we have $\lambda_{M\backslash a}(S_i) = 2$ and $|S_i - S_{i-1}| = 1$. Each member of the corresponding sequence in N is certainly 3-separating, and, after ignoring possible equal members of this sequence, we obtain a sequence of 3-separating sets in N that guarantees the truth of 10.4.2.

Using 10.4.2 and complements, we deduce from Lemma 10.4 that $\mathbf{Q}$ is a tight flower equivalent to $\mathbf{R}$. By using the fact that $\mathbf{R}$ satisfies (i)–(iii) of the corollary, we conclude that the assertions in the first sentences of (i)–(iii) hold for **Q**. The assertions in the second sentences follow by applying the dual of Lemma 10.1 to **Q**. □

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